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Optimal systems of Lie subalgebras for a two-phase mass flow

Sayonita GhoshHajra[a,*], Santosh Kandel[b], Shiva P. Pudasaini[c]

[a] Department of Mathematics, Hamline University, 1536 Hewitt Avenue, MS-B1807, Saint Paul, MN 55104, USA
[b] Institute für Mathematik, University of Zürich, Winterthurerstrasse 190, CH-8057, Zürich
[c] Department of Geophysics, Steinmann Institute, University of Bonn, Meckenheimer Allee 176, D-53115, Bonn, Germany

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ABSTRACT

We apply the Lie symmetry method to a two-phase mass flow model (Pudasaini, 2012 [18]) and construct one-, two- and three-dimensional optimal systems of Lie subalgebras corresponding to the non-linear PDEs. As an optimal system contains structurally important information about different types of invariant solutions, it provides precise insights into all possible invariant solutions emerging from infinitesimal symmetries. We use the optimal system of one-dimensional Lie subalgebras to reduce the two-phase mass flow model to other systems of PDEs. Using the fact that the Lie bracket contains information about further reduction, we further reduce to systems of ODEs and PDEs. We solve a system numerically and present results for different physical and Lie parameters. Simulations reveal fluid and solid dynamics are distinctly sensitive to different Lie parameters, whereas both phases are influenced by the solid and the fluid pressure parameters. Higher pressure gradients result in higher flow velocities and lower flow heights. Fluid velocities dominate solid velocities, but the solid heights are higher than the fluid heights. Results provide an overall picture of the physical process, and the coupled dynamics of the solid and fluid phase velocities and the flow heights. These are physically meaningful results in sheared inclined channel flow of coupled two-phase mixture. This confirms the consistency of the obtained similarity solutions and potential applicability of the models and the constructed optimal systems.

1. Introduction

Rapid gravity mass flows and gravity currents such as debris flows, debris floods, and water waves are common natural phenomena. Due to their complex nature these events pose greater challenges to environmental, geophysical and engineering communities [4–7,10,16,17,21,22,31]. In this paper, we are mainly concerned with debris flows, which are effectively two-phase gravity-driven mass flows consisting of a mixture of solid particles and viscous fluid. These flows are mainly described by the relative motion between the solid and the fluid phases. Their evolution primarily depends on the solid–fluid mixture composition, mixture interactions, and mixture dynamics modelled by the driving forces. The debris flows are extremely destructive natural hazards. Hence, we need a reliable way to predict the dynamics of the flow. There has been extensive (field, experimental, theoretical, and numerical) studies investigating the dynamics, and consequences of these flows including their industrial applications [9,21,25,26,28]. In [18] Pudasaini presented a new theory, which describes the different interactions between the solid and the fluid in two-phase debris flows including several fundamentally important and dominant physical aspects which constitutes the most generalized two-phase flow model, as a set of partial differential equations in the conservative form [18,19,21] available to date. This theory supports the strong interactions between the solid and the fluid phases.

In this paper, we apply the Lie symmetry method to the two-phase mass flow model [18] and construct optimal systems of subalgebras corresponding to this system of non-linear PDEs. Lie symmetry method is an important tool for examining differential equations and constructing their solutions [1,8]. The heart of this method lies in the construction of the invariant solutions, the solutions which are invariant under the Lie subgroups of the Lie group of symmetries, and the construction of new solutions from known solutions by using symmetries. Invariant solutions are constructed by reducing the given system of PDEs into another system of PDEs or ODEs with the aid of Lie subgroups and solving the reduced system. Since almost every Lie group of symmetries has infinitely many Lie subgroups, it seems the quest for invariant solutions is a very challenging task. As a symmetry transformation maps one solution to another, practically, it is sufficient to find the invariant solutions which are not related by any member of the Lie group of symmetries. This suggests to look for ways to

* Corresponding author.
E-mail address: sghoshhajra01@hamline.edu (S. GhoshHajra).

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determine Lie subgroups which generates fundamentally distinct invariant solutions. It is of great importance from the mathematical point of view as well as constraining the physical and engineering applications, as it provides precise insights into all possible invariant solutions of the system. This problem can be solved by classifying the Lie subgroups of the Lie group of symmetries, often called the optimal system of s-parameter subgroups [11,12]. It possesses an infinitesimal counterpart called the optimal system of s-parameter subalgebras.

In the literature, mostly, single-phase, inviscid and pressure-driven shallow water flows in horizontal channels with no friction models are investigated. For example, in [15] similarity forms for two-layer shallow water equations are derived and discussed where the coupling is via the reduced gravity parameter. In [1], GhoshHajra et al. discuss a two-phase mass flow model whose driving forces consist of gravity, buoyancy, and hydraulic pressure gradients for the solid and fluid phases respectively. The two-phase mass flow model utilized there is derived from a general two-phase mass flow model [18] as a mixture of sediment particles and viscous fluid with strong non-linear phase interactions. These interactions pose great challenges in constructing exact solutions as compared to the effectively single-phase gravity mass flows. Thus, the problem, which is the study of the phase interaction between the solid and the fluid in a two-phase mass flow, is highly non-trivial and novel.

Using an elementary Lie transformation of the model [18], GhoshHajra et al. [1] constructed several similarity forms and similarity variables, which generalizes several similarity variables and similarity forms obtained in the literature by using Lie symmetry group transformations [2,3,13–15,23,24,27,30]. They also constructed several reduced homogeneous and non-homogeneous systems of ODEs and provided some analytical and numerical solutions. Here, we consider the simplified model from [1] and advance further by constructing their optimal systems of Lie subalgebras and providing simulations and applications. For this, we first briefly discuss the two-phase mass flow model and the symmetry Lie algebra of the given system of PDEs from [1]. Then, we construct the optimal system of Lie subalgebras of dimensions one, two, and three. Further, we analyze in detail the reduced system of PDEs obtained using the one-dimensional Lie subalgebras. We construct several reduced systems of ODEs and PDEs. We also present simulation results for some systems in order to have an in-depth and quantitative analyses of these systems. This allows for the analysis of impact of the model parameters in the physical–mathematical model [18] and the Lie parameters in the optimal system under consideration. As all the reduced ODE systems obtained here are non-homogeneous, these systems are more complex both from the mathematical structure and the underlying physics than the systems presented and discussed in [1]. Consequently, the new systems potentially cover wider spectrum of applications.

2. The two-phase mass flow model and the symmetry Lie algebra

We consider the general two-phase debris flow model [18] that was reduced to one-dimensional inclined channel flow [21]. For completeness, we briefly recall the basic features of the model equations [1].[1] The depth-averaged mass and momentum conservation equations for the solid and fluid phases are as follows [18]:

$$\frac{\partial h_s}{\partial t} + \frac{\partial Q_s}{\partial X} = 0, \quad \frac{\partial h_f}{\partial t} + \frac{\partial Q_f}{\partial X} = 0, \tag{1}$$

$$\begin{aligned} &\frac{\partial Q_s}{\partial t} + \frac{\partial}{\partial X}(Q_s^2 h_s^{-1}) + \frac{\partial}{\partial X}\left(\frac{\beta_s}{2} h_s (h_s + h_f)\right) = h_s S_s,\\ &\frac{\partial Q_f}{\partial t} + \frac{\partial}{\partial X}(Q_f^2 h_f^{-1}) + \frac{\partial}{\partial X}\left(\frac{\beta_f}{2} h_f (h_f + h_s)\right) = h_f S_f, \end{aligned} \tag{2}$$

where S_s, S_f are the solid and fluid net driving forces, given by $S_s = \sin\zeta - \tan\delta(1-\gamma)\cos\zeta,\ S_f = \sin\zeta$. The dynamical variables and parameters are

$$h_s = \alpha_s h,\ h_f = \alpha_f h;\ Q_s = h_s u_s = \alpha_s h u_s,\ Q_f = h_f u_f = \alpha_f h u_f;\ \beta_s = \varepsilon K p_{b_s},$$
$$\beta_f = \varepsilon p_{b_f},\ p_{b_f} = \cos\zeta,\ p_{b_s} = (1-\gamma) p_{b_f},\ \alpha_f = 1 - \alpha_s,\ \gamma = \frac{\rho_f}{\rho_s}.$$

Here, t is the time, X and Z are coordinates along and normal to the slope with inclination ζ. The solid particles and fluid constituents are denoted by the suffices s and f respectively. The mixture flow depth is h, and the solid and fluid velocities are u_s and u_f respectively. The densities and volume fractions are ρ_s, ρ_f, and α_s, α_f respectively, L and H denote the typical length and depth of the flow with the aspect ratio $\varepsilon = H/L$. Both K, the earth pressure coefficient, and $\tan\delta$, where δ is the friction angle, include frictional behavior of the solid-phase. Here p_{b_f} and p_{b_s} are associated with the effective basal fluid and solid pressures, β_s, β_f are the hydraulic pressure parameters associated with the solid- and the fluid-phases respectively, and γ is the density ratio, $(1-\gamma)$ indicates the buoyancy reduced solid normal load. The solid and fluid fractions in the mixture are h_s, h_f, and the solid and fluid fluxes are Q_s, Q_f.

For convenience, GhoshHajra et al. [1] introduced the transformation

$$x = X - S_s t^2/2, \quad y = X - S_f t^2/2, \ \widehat{Q}_s = Q_s - h_s S_s t, \quad \widehat{Q}_f = Q_f - h_f S_f t. \tag{3}$$

Here, x and y are the moving spatial coordinates for the solid and fluid respectively. Since S_s and S_f are independent, x and y are considered as independent variables [1]. Using (3), Eqs. (1)–(2) become a homogeneous system of partial differential equation:

$$\frac{\partial h_s}{\partial t} + \frac{\partial \widehat{Q}_s}{\partial x} = 0, \ \frac{\partial h_f}{\partial t} + \frac{\partial \widehat{Q}_f}{\partial y} = 0, \tag{4}$$

$$\begin{aligned} &\frac{\partial \widehat{Q}_s}{\partial t} + \frac{\partial}{\partial x}(\widehat{Q}_s^2 h_s^{-1}) + \frac{\partial}{\partial x}\left(\frac{\beta_s}{2} h_s (h_s + h_f)\right) = 0,\\ &\frac{\partial \widehat{Q}_f}{\partial t} + \frac{\partial}{\partial y}(\widehat{Q}_f^2 h_f^{-1}) + \frac{\partial}{\partial y}\left(\frac{\beta_f}{2} h_f (h_f + h_s)\right) = 0. \end{aligned} \tag{5}$$

Replacing $\widehat{Q}_s = \hat{u}_s h_s$, $\widehat{Q}_f = \hat{u}_f h_f$, introducing the suffices $1{:=}s$, $2{:=}f$ for the variables and parameters associated with the solid and fluid components respectively, and dropping the hats, (4)–(5) changes to

$$\begin{aligned} &\frac{\partial h_1}{\partial t} + \frac{\partial (u_1 h_1)}{\partial x} = 0,\\ &\frac{\partial h_2}{\partial t} + \frac{\partial (u_2 h_2)}{\partial y} = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} &\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \beta_1 \frac{\partial h_1}{\partial x} + \frac{\beta_1}{2}\frac{\partial h_2}{\partial x} + \frac{\beta_1}{2}\frac{h_2}{h_1}\frac{\partial h_1}{\partial x} = 0,\\ &\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial y} + \beta_2 \frac{\partial h_2}{\partial y} + \frac{\beta_2}{2}\frac{\partial h_1}{\partial y} + \frac{\beta_2}{2}\frac{h_1}{h_2}\frac{\partial h_2}{\partial y} = 0. \end{aligned} \tag{7}$$

As discussed in [1], the mass flow model (6)–(7) which is deduced from the mixture model [18] is different from the other existing models. The fourth and fifth terms associated with $\beta/2$ in (7) that emerge from the pressure-gradients, include buoyancy (through $1-\gamma$), friction (through K and $\tan\delta$), net driving forces (S_s, S_f) and the coordinate transformations (3) that incorporate gravity, friction, and buoyancy. These forces have mechanical significance in explaining the physics of the two-phase gravity mass flows that are not discussed in previous models, and in

[1] This discussion follows [1, pp. 326–327].

Table 1
Lie brackets for the generators $\{V_1, V_2, V_3, V_4, V_5\}$ of the Lie algebra $\mathfrak{g}$.

[,]	V_1	V_2	V_3	V_4	V_5
V_1	0	0	0	V_1	0
V_2	0	0	0	V_2	0
V_3	0	0	0	0	V_3
V_4	$-V_1$	$-V_2$	0	0	0
V_5	0	0	$-V_3$	0	0

[1], they are considered for the mixture flows.

GhoshHajra et al. [1] initiated the use of Lie symmetry method to study the system (6)–(7). They computed the most general symmetry Lie algebra $\mathfrak{g}$ of these systems of PDEs, and applied the symmetries to reduce the system into systems of ODEs. The Lie algebra $\mathfrak{g}$ is five dimensional with a basis $\{V_1, V_2, V_3, V_4, V_5\}$, where

$$V_1=\frac{\partial}{\partial x},\quad V_2=\frac{\partial}{\partial y},\quad V_3=\frac{\partial}{\partial t},$$
$$V_4=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+2h_1\frac{\partial}{\partial h_1}+2h_2\frac{\partial}{\partial h_2}+u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2},$$
$$V_5=t\frac{\partial}{\partial t}-2h_1\frac{\partial}{\partial h_1}-2h_2\frac{\partial}{\partial h_2}-u_1\frac{\partial}{\partial u_1}-u_2\frac{\partial}{\partial u_2}.$$

The Lie bracket operation on the basis $\{V_1, V_2, V_3, V_4, V_5\}$ is shown in Table 1.

3. Optimal system of Lie subalgebras

Generally, it is difficult to gain insight on all the possible invariant solutions as there can be infinitely many Lie subgroups of the Lie group of symmetries G of a given system. Hence, it is desirable to partition all the possible invariant solutions into disjoint sets so that two solutions belonging to the same set are similar (one solution can be transformed into the other by an element of the symmetry Lie group) and belonging to the different sets are not similar (these solutions are not related by any member of the symmetry Lie group). This classification problem has a solution, namely the construction of an optimal system of s-parameter subgroups [11,12]. Often, it is easier to work with the symmetry Lie algebra, i.e., construction of optimal systems of s-parameter Lie subalgebras of the symmetry Lie algebra, and in nice situations, an optimal system of s-dimensional Lie subalgebras is sufficient to construct an optimal system of s-parameter subgroups.

An optimal system of s-dimensional Lie subalgebras (s-parameter subgroups) contains structurally important information about different types of invariant solutions [11,12]. Let $\mathfrak{g}$ be the Lie algebra of G. An optimal system of s-parameter Lie subalgebras is a list of s-dimensional Lie subalgebras of $\mathfrak{g}$ such that every s-dimensional Lie subalgebra of $\mathfrak{g}$ is equivalent to a unique member of the list and no two Lie subalgebras in the list are equivalent to each other. Two s-dimensional Lie subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ are equivalent if there is an element g of G such that $\mathfrak{h}'=\mathrm{Ad}_g(\mathfrak{h})$ where Ad_g is the adjoint action of g on $\mathfrak{g}$. We can use the exponential map Exp from $\mathfrak{g}$ to G [11,12] to define the adjoint action of a generic $g\in G$.

Assume that $g=\mathrm{Exp}(V)$ for some $V\in\mathfrak{g}$. Note that any $V\in\mathfrak{g}$ defines a linear operator $\mathrm{ad}(V):\mathfrak{g}\to\mathfrak{g}$ given by

$$\mathrm{ad}(V)(W)=[V, W]$$

where [,] is the Lie bracket. Now, represent $\mathrm{ad}(V)$ by a matrix by choosing a basis of $\mathfrak{g}$. Then (see [11]),

$$\mathrm{Ad}_g=e^{\mathrm{ad}(V)}=\sum_{n=0}^{\infty}\frac{\mathrm{ad}(V)^n}{n!}. \tag{8}$$

Let $\mathfrak{g}$ be the symmetry Lie algebra with basis $\{V_1, V_2, V_3, V_4, V_5\}$ of Section 2 and identify this with $\mathbb{R}^5$ as a vector space using the map $V_i \mapsto e_i$ where $\{e_1,\dots,e_5\}$ is the standard basis of $\mathbb{R}^5$. Then, from Table 1, we get the following matrix representations of $\mathrm{ad}(V_i)$:

$$\mathrm{ad}(V_1)=\begin{bmatrix}0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},\quad \mathrm{ad}(V_2)=\begin{bmatrix}0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},$$
$$\mathrm{ad}(V_3)=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},\quad \mathrm{ad}(V_4)=\begin{bmatrix}-1&0&0&0&0\\0&-1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},$$
$$\mathrm{ad}(V_5)=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&-1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}. \tag{9}$$

Let ε_i, $i=1,2,\dots,5$, be real constants and $g_i=\mathrm{Exp}(\varepsilon_i V_i)$. Then, by computing the exponential of matrices $\varepsilon_i\mathrm{ad}(V_i)$, we get

$$\mathrm{Ad}_{g_1}=\begin{bmatrix}1&0&0&\varepsilon_1&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},\quad \mathrm{Ad}_{g_2}=\begin{bmatrix}1&0&0&0&0\\0&1&0&\varepsilon_2&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},$$
$$\mathrm{Ad}_{g_3}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&\epsilon_3\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},\quad \mathrm{Ad}_{g_4}=\begin{bmatrix}e^{-\varepsilon_4}&0&0&0&0\\0&e^{-\varepsilon_4}&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},$$
$$\mathrm{Ad}_{g_5}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&e^{-\varepsilon_5}&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}. \tag{10}$$

The adjoint action Ad_{g_i} of g_i on the basis $\{V_1,\dots,V_5\}$ of $\mathfrak{g}$ is summarized in Table 2.

Construction of optimal systems of s-parameter subalgebras of a Lie algebra $\mathfrak{g}$ is a difficult task. To our knowledge, there is no systematic way to compute an optimal system when $s>3$. However, when $s\le 3$, we can compute optimal systems of s-parameter subalgebras. The Lie subalgebras can be computed by solving some algebraic equations, and the equivalent Lie subalgebras can be identified by applying the adjoint action Ad_g on the set of these Lie subalgebras [12]. In connection to the coupled system of two-phase mass flows (6)–(7), this is one of the contributions of this paper. Here, we construct one-, two-, and three-dimensional optimal systems of Lie subalgebras for the physical system (6)–(7).

3.1. Optimal system of one-dimensional Lie subalgebras

Since any nonzero element of $\mathfrak{g}$ generates one dimensional Lie subalgebra, it is possible to identify whether two one-dimensional Lie subalgebras are equivalent by studying whether their generators are related by the adjoint action Ad_g for some g of the symmetry Lie group. In other words, it is sufficient to concentrate on how a general Ad_g transforms a general element of $\mathfrak{g}$ and use that information to identify

Table 2
The adjoint action Ad_{g_i}, $i=1,2,\dots,5$, on the basis $\{V_1, V_2, V_3, V_4, V_5\}$.

Ad	V_1	V_2	V_3	V_4	V_5
g_1	V_1	V_2	V_3	$V_4+\varepsilon_1 V_1$	V_5
g_2	V_1	V_2	V_3	$V_4+\varepsilon_2 V_2$	V_5
g_3	V_1	V_2	V_3	V_4	$V_5+\varepsilon_3 V_3$
g_4	$e^{-\varepsilon_4}V_1$	$e^{-\varepsilon_4}V_2$	V_3	V_4	V_5
g_5	V_1	V_2	$e^{-\varepsilon_5}V_3$	V_4	V_5

equivalent Lie subalgebras.Let $V = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5$ be a general element of $\mathfrak{g}$ where a_i, $i = 1, 2, \ldots, 5$, are real constants. We can think of V as a column vector with entries $a_1, \ldots, a_5$. Let $A(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) = \mathrm{Ad}_{g_5}\circ\mathrm{Ad}_{g_4}\circ\mathrm{Ad}_{g_3}\circ\mathrm{Ad}_{g_2}\circ\mathrm{Ad}_{g_1}$, then multiplying the matrices in (10) we get

$$A(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) = \begin{bmatrix} e^{-\varepsilon_4} & 0 & 0 & \varepsilon_1 e^{-\varepsilon_4} & 0 \\ 0 & e^{-\varepsilon_4} & 0 & \varepsilon_2 e^{-\varepsilon_4} & 0 \\ 0 & 0 & e^{-\varepsilon_5} & 0 & \varepsilon_3 e^{-\varepsilon_5} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

Moreover, $A(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)$ transforms V as follows:

$$A(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)V = (a_1 + \varepsilon_1 a_4)e^{-\varepsilon_4}V_1 + (a_2 + \varepsilon_2 a_4)e^{-\varepsilon_4}V_2 + (a_3 + \varepsilon_3 a_5)e^{-\varepsilon_5}V_3 + a_4V_4 + a_5V_5. \tag{12}$$

By definition V and $A(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)V$ generate equivalent one-dimensional Lie subalgebras for any $\varepsilon_1, \ldots, \varepsilon_5$. This gives the freedom of choosing various values of ε_i so that a representative of the equivalence class of V might be much simpler than V. The following lemmas will be useful for this purpose.

Lemma 3.1. *When* $a_4 \neq 0$, $a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5$ *and* $a_3V_3 + a_4V_4 + a_5V_5$ *generate equivalent Lie subalgebras.*

Proof. Choosing $\varepsilon_1 = -\frac{a_1}{a_4}$, $\varepsilon_2 = -\frac{a_2}{a_4}$ and $\varepsilon_3 = 0$, we see that $a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5$ and $V_4 + a_3e^{-\varepsilon_5}V_3 + a_5V_5$ generate equivalent one-dimensional Lie subalgebras.□Lemma 3.1 says the equivalence class of one-dimensional Lie subalgebras containing the Lie subalgebra $\langle a_1V_1 + a_2V_2 + a_3V_3 + V_4 + a_5V_5\rangle$ is independent of a_1 and a_2. The key reasons behind this lemma are the Lie bracket relations $[V_4, V_1] = -V_1$ and $[V_4, V_2] = -V_2$, which is reflected in Ad_{g_1}, Ad_{g_2} and Ad_{g_4}.

Lemma 3.2. *The Lie subalgebra* $\langle a_3V_3 + V_4 + a_5V_5\rangle$ *is equivalent to exactly one of the Lie subalgebras* $\langle V_4\rangle$, $\langle V_4 + a_5V_5\rangle$, $\langle V_3 + V_4 + a_5V_5\rangle$ *and* $\langle -V_3 + V_4 + a_5V_5\rangle$.

Proof. When $a_3 = 0 = a_5$, the Lie subalgebra is $\langle V_4\rangle$.When $a_3 = 0$, the Lie subalgebra is $\langle V_4 + a_5V_5\rangle$.

When $a_3 \neq 0$, taking $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$, we observe

$$A(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)(a_3V_3 + V_4 + a_5V_5) = a_3\varepsilon^{-\varepsilon_5}V_3 + V_4 + a_5V_5. \tag{13}$$

Choosing ε_5 so that either $a_3e^{-\varepsilon_5} = 1$ or $a_3e^{-\varepsilon_5} = -1$, it follows that the Lie subalgebra $\langle a_3V_3 + V_4 + a_5V_5\rangle$ is equivalent to $\langle V_3 + V_4 + a_5V_5\rangle$ or $\langle -V_3 + V_4 + a_5V_5\rangle$. This proves the lemma.□

Lemmas 3.1 and 3.2 imply the Lie subalgebra $\langle a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5\rangle$, $a_4 \neq 0$, is equivalent to one of the following types of one-dimensional Lie subalgebras

$$\langle V_4\rangle, \langle V_4 + k_1V_5\rangle, \langle V_3 + V_4 + k_2V_5\rangle, \langle -V_3 + V_4 + k_3V_5\rangle$$

where $k_1 \neq 0$, k_2 and k_3 are constants.

Similar analysis can be carried out for $a_5 \neq 0$, it can be shown $\langle a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5\rangle$ and $\langle a_1V_1 + a_2V_2 + a_4V_4 + a_5V_5\rangle$ generate equivalent one-dimensional Lie subalgebras. Similarly, $\langle a_1V_1 + a_2V_2 + a_3V_3 + a_5V_5\rangle$ is equivalent to one of the following:

$$\langle V_1 + k_4V_2 + V_5\rangle, \langle -V_1 + k_5V_2 + V_5\rangle, \langle k_6V_1 + V_2 + V_5\rangle, \langle k_7V_1 - V_2 + V_5\rangle$$

where k_4, k_5, k_6 and k_7 are constants. Analyzing the rest of the cases in the similar way we get:

Proposition 3.3. *An optimal system of one-dimensional Lie subalgebras contains Lie subalgebras*

$$\langle V_1\rangle, \langle V_3\rangle, \langle V_5\rangle, \langle V_3 + V_4\rangle, \langle -V_3 + V_4\rangle,$$

and the following families of subalgebras

$\{\langle k_1V_1 + V_2\rangle: k_1 \in \mathbb{R}\}$, $\{\langle V_4 + k_2V_5\rangle: k_2 \in \mathbb{R}\}$,
$\{\langle V_1 + k_3V_2 + V_5\rangle: k_3 \in \mathbb{R}\}$, $\{\langle -V_1 + k_4V_2 + V_5\rangle: k_4 \in \mathbb{R}\}$,
$\{\langle V_1 + k_5V_2 + V_3\rangle: k_5 \in \mathbb{R}\}$, $\{\langle V_1 + k_6V_2 - V_3\rangle: k_6 \in \mathbb{R}\}$,
$\{\langle k_7V_1 + V_2 + V_5\rangle: k_7 \in \mathbb{R}\setminus\{1, -1\}\}$, $\{\langle k_8V_1 - V_2 + V_5\rangle: k_8 \in \mathbb{R}\setminus\{1, -1\}\}$,
$\{\langle k_9V_1 + V_2 + V_3\rangle: k_9 \in \mathbb{R}\setminus\{1, -1\}\}$, $\{\langle k_{10}V_1 + V_2 - V_3\rangle: k_{10} \in \mathbb{R}\setminus\{1, -1\}\}$.

3.2. Optimal system of two-dimensional Lie subalgebras

Next, we compute an optimal system of two-dimensional Lie subalgebras. The key idea is to analyze only those two-dimensional Lie subalgebras which are extensions of the one-dimensional Lie subalgebras from the list in Proposition 3.3 rather than all the two-dimensional Lie subalgebras [12]. This suggests, we first compute all the two-dimensional Lie subalgebras which are extensions of the one-dimensional Lie subalgebras from the list in Proposition 3.3, then, remove the redundancies by studying action of a general Ad_g on these Lie subalgebras.

The two-dimensional Lie subalgebras which are extension of one-dimensional Lie subalgebras from Proposition 3.3 can be computed as follows. Let $\langle W\rangle$ be a one-dimensional Lie subalgebra from Proposition 3.3 and consider the problem of finding all two-dimensional Lie subalgebras containing $\langle W\rangle$. To solve this problem, we must find all possible $V \in \mathfrak{g}$ such that $\langle V, W\rangle$ is a two-dimensional Lie subalgebra. As we want $\langle V, W\rangle$ to be a two-dimensional Lie subalgebra, there must be constants λ and μ such that

$$[W, V] = \lambda W + \mu V. \tag{14}$$

In summary, we must find V such that $\langle V, W\rangle$ is a two dimensional vector space, and λ and μ satisfy (14). This will lead to a set of algebraic equations whose solutions will give the two-dimensional Lie subalgebras containing $\langle W\rangle$.

To illustrate, we compute all two-dimensional Lie subalgebras which contain the one-dimensional Lie subalgebra $\langle V_4\rangle$. Let $V = a_1V_1 + a_2V_2 + a_3V_3 + a_5V_5$ be a second basis element of a desired two-dimensional Lie subalgebra. Table 1 gives $[V_4, V] = -a_1V_1 - a_2V_2$, which provides the left hand side of Eq. (14). The problem, here, is to find a_1, a_2, a_3, a_5 not simultaneously zero and λ and μ such that

$$\mu a_1 = -a_1, \quad \mu a_2 = -a_2, \quad \mu a_3 = 0,$$

$$\lambda = 0, \quad \mu a_5 = 0.$$

Case 1: When $\mu = 0$ then, $a_1 = a_2 = 0$. Any a_3 and a_5, not simultaneously zero, can be chosen arbitrarily. This results in the two-dimensional Lie subalgebras $\langle a_3V_3 + a_5V_5, V_4\rangle$, $(a_3, a_4) \in \mathbb{R}^2\setminus\{(0, 0)\}$.

Case 2: When $\mu \neq 0$ then, $a_3 = a_5 = 0$. As $V \neq 0$, $\mu = -1$ is the only choice and any a_1 and a_2, not simultaneously zero can be chosen as free parameters. This results in another two-dimensional Lie subalgebras $\langle a_1V_1 + a_2V_2, V_4\rangle$, $(a_1, a_2) \in \mathbb{R}^2\setminus\{(0, 0)\}$.

This process, applied to all the one-dimensional Lie subalgebras from Proposition 3.3, computes the two-dimensional Lie subalgebras which contain one-dimensional Lie subalgebras from Proposition 3.3. Next, we must eliminate redundancies by studying equivalence classes of these Lie subalgebras to compute an optimal system. We achieve this by analyzing how these Lie subalgebras transform when a general Ad_g applied to them.

To demonstrate this, we compute the equivalence class of the subalgebra $\langle a_3V_3 + a_5V_5, V_4\rangle$ for fixed $(a_3, a_5) \in \mathbb{R}^2\setminus\{(0, 0)\}$ and find a simplified representative of the equivalence class.

Let $A = A(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)$ be as in (12), then

$$A(a_3V_3 + a_5V_5) = e^{-\varepsilon_5}(a_3 + a_5\varepsilon_3)V_3 + a_5V_5 \quad \text{and}$$

$$A(V_4) = \varepsilon_1 e^{-\varepsilon_4}V_1 + \varepsilon_2 e^{-\varepsilon_4}V_2 + V_4.$$

This means, by definition, $\langle a_3V_3 + a_5V_5, V_4\rangle$ and $\langle e^{-\varepsilon_5}(a_3 + a_5\varepsilon_3)V_3 + a_5V_5, \varepsilon_1 e^{-\varepsilon_4}V_1 + \varepsilon_2 e^{-\varepsilon_4}V_2 + V_4\rangle$ are equivalent Lie subalgebras. Following this process, we obtain all two-dimensional

Lie subalgebras from the two cases (*Case* 1, *Case* 2) above which are equivalent to $\langle a_3V_3 + a_5V_5, V_4\rangle$ by choosing various values of ε_i's. The following lemma is useful to determine the equivalence classes of two-dimensional subalgebras.

Lemma 3.4. *For all* $a_5 \neq 0$, $\langle a_3V_3 + a_5V_5, V_4\rangle$ *is equivalent to* $\langle V_5, V_4\rangle$.

Proof. Take $\varepsilon_1 = \varepsilon_2 = \varepsilon_5 = 0$ and ε_3 such that $a_3 + a_5\varepsilon_3 = 0$. Then, $A(a_3V_3 + a_5V_5) = a_5V_5$. Hence, for this particular A, we have $A(\langle a_3V_3 + a_5V_5, V_4\rangle) = \langle V_4, a_5V_5\rangle = \langle V_4, V_5\rangle$. □

Lemma 3.4 says $\langle V_4, V_5\rangle$ can be taken as a simple representative of the equivalence class of the Lie subalgebras $\langle a_3V_3 + a_5V_5, V_4\rangle$ when $a_5 \neq 0$.

Corollary 3.5. *We can take* $\langle V_3, V_4\rangle$ *and* $\langle V_4, V_5\rangle$ *as the representative of equivalence classes of Lie subalgebras* $\langle a_3V_3 + a_5V_5, V_4\rangle$ *for any* $(a_3, a_5) \in \mathbb{R}^2\setminus\{(0, 0)\}$.We consider only one representative from each equivalence class to get an optimal system. The simplest choices are $\langle V_3, V_4\rangle$ and $\langle V_4, V_5\rangle$ for the Lie subalgebras $\langle a_3V_3 + a_5V_5, V_4\rangle$ for $(a_3, a_5) \in \mathbb{R}^2\setminus\{(0, 0)\}$.

A similar analysis to all the other two-dimensional Lie subalgebras containing one-dimensional Lie subalgebras from Proposition 3.3 leads to the following conclusion.

Proposition 3.6. *An optimal system of the two-dimensional Lie subalgebras can be represented by the following Lie subalgebras*:

$\langle V_1, V_2\rangle, \langle V_1, V_3\rangle, \langle V_1, V_5\rangle, \langle V_3, V_4\rangle, \langle V_4, V_5\rangle$

and the families of Lie subalgebras

$\{\langle k_1V_1 + V_2, a_1V_1 + V_3\rangle: (k_1, a_1) \in \mathbb{R}^2\}$,
$\{\langle k_1V_1 + V_2, a_2V_1 + V_5\rangle: (k_1, a_2) \in \mathbb{R}^2\}$,
$\{\langle k_1V_1 + V_2, a_3V_3 + V_4\rangle: (k_1, a_3) \in \mathbb{R}^2\}$,
$\{\langle -V_1 - V_2 + V_3, a_4V_1 + V_2\rangle: a_4 \in \mathbb{R}\}$,
$\{\langle V_1, a_5V_3 + V_4\rangle: a_5 \in \mathbb{R}\}$,
$\{\langle V_4 + k_3V_5, a_6V_1 + a_7V_2 + V_3\rangle: (a_6, a_7) \in \mathbb{R}^2$,
$k_3 \in \mathbb{R}\setminus\{0\}\}$, $\{\langle V_1, V_4 + k_3V_5\rangle: k_3 \in \mathbb{R}\setminus\{0\}\}$,
$\{\langle V_4 + k_3V_5, a_8V_1 + V_2\rangle: k_3 \in \mathbb{R}\setminus\{0\}, a_8 \in \mathbb{R}\}$,
$\{\langle V_1 + k_4V_2 + V_5, V_1 + a_9V_2\rangle: (k_4, a_9) \in \mathbb{R}^2\}$,
$\{\langle -V_1 + k_5V_2 + V_5, V_1 + a_{10}V_2\rangle: (k_5, a_{10}) \in \mathbb{R}^2\}$,
$\{\langle V_1 + k_6V_2 + V_3, V_1 + a_{11}V_2\rangle: (k_6, a_{11}) \in \mathbb{R}^2\}$,
$\{\langle V_1 + k_7V_2 - V_3, a_{12}V_2 + V_3\rangle: (k_7, a_{12}) \in \mathbb{R}^2\}$,
$\{\langle k_8V_1 + V_2 + V_5, a_{13}V_1 + V_2\rangle: (k_8, a_{13}) \in \mathbb{R}^2\}$,
$\{\langle k_9V_1 - V_2 + V_5, a_{14}V_1 + V_2\rangle: (k_9, a_{14}) \in \mathbb{R}^2\}$,
$\{\langle k_{10}V_1 + V_2 + V_3, a_{15}V_1 + V_3\rangle: (k_{10}, a_{15}) \in \mathbb{R}^2\}$,
$\{\langle V_3, a_{17}V_1 + a_{18}V_2 + V_5\rangle: (a_{17}, a_{18}) \in \mathbb{R}^2\}$,
$\{\langle k_{11}V_1 + V_2 - V_3, a_{16}V_2 + V_3\rangle: a_{16} \neq -1, k_{11} \neq 0\}$.

3.3. Optimal system of three-dimensional Lie subalgebras

The construction of an optimal system of three-dimensional Lie subalgebras is exactly similar to construction of an optimal system of two-dimensional Lie subalgebras in the previous subsection: First, extend the two-dimensional Lie subalgebras from the proposition (Proposition 3.6) to three-dimensional Lie subalgebras in all possible ways, and then, remove all the redundancies by applying a general Ad_g [12]. The only difference is, here, there are more algebraic equations. The result is summarized in:

Proposition 3.7. *The following subalgebras form an optimal system of three-dimensional Lie subalgebras*:

$\langle V_1, V_2, CV_4 + V_5\rangle, \langle V_1, V_2, CV_3 + V_4\rangle, \langle V_1, V_2, V_3\rangle$,
$\langle V_3, V_4, CV_1 + V_2\rangle, \langle V_3, V_4, V_5\rangle, \langle V_1, V_3, V_4 + CV_5\rangle$,
$\langle V_2, V_4, V_5\rangle, \{\langle k_1V_1 + V_2, V_3, V_4 + CV_5\rangle, k_1 \in \mathbb{R}\}$,
$\{\langle k_1V_1 + V_2, V_3, CV_1 + V_5\rangle, k_1 \in \mathbb{R}\}$,
$\{\langle -V_1 - V_2 + V_3, a_4V_1 + V_2, V_4 + V_5\rangle, a_4 \neq 1\}$,
$\{\langle V_1, a_5V_3 + V_4, V_5\rangle, a_5 \in \mathbb{R}\}$, $\{\langle V_4 + k_3V_5, a_6V_1 + a_7V_2 + V_3, CV_1 + V_2\rangle$,
$k_3 \neq 0, (a_6, a_7) \in \mathbb{R}^2\}$, $\{\langle V_4 + k_3V_5, a_7V_2 + V_3, V_1\rangle, k_3 \neq 0\}$,
$\{\langle V_4 + V_5, a_8V_1 + V_2, CV_1 + V_3\rangle, a_8 \in \mathbb{R}\}$, $\{\langle V_5, V_1 + k_4V_2, V_4\rangle, k_4 \in \mathbb{R}\}$,
$\{\langle V_1 + k_4V_2 + V_5, V_1 + a_9V_2, V_3\rangle, (k_4, a_9) \in \mathbb{R}^2\}$,
$\{\langle -V_1 + k_5V_2 + V_5, V_1 + a_{10}V_2, V_3\rangle, (k_5, a_{10}) \in \mathbb{R}^2\}$,
$\{\langle V_1 + k_6V_2 + V_3, V_1 + a_{11}V_2, V_4 + V_5\rangle, k_6 \neq a_{11}\}$,
$\{\langle V_3, V_1 + k_6V_2, CV_2 + V_5\rangle, k_6 \in \mathbb{R}\}$,
$\{\langle V_1 + k_7V_2 - V_3, a_{12}V_2 + V_3, V_4 + V_5\rangle, k_7 \in \mathbb{R}, a_{12} \neq 0\}$,
$\{\langle k_8V_1 + V_2 + V_5, a_{13}V_1 + V_2, V_3\rangle, (k_8, a_{13}) \in \mathbb{R}^2\}$,
$\{\langle V_3, a_{17}V_1 + a_{18}V_2 + V_5, CV_1 + V_2\rangle$,
$(a_{17}, a_{18}) \in \mathbb{R}^2\setminus(0, 0)\}\{\langle k_9V_1 - V_2 + V_5, a_{14}V_1 + V_2, V_3\rangle, (k_9, a_{14}) \in \mathbb{R}^2\}$,
$\{\langle k_{10}V_1 + V_2 + V_3, a_{15}V_1 + V_3, V_4 + V_5\rangle, a_{15} \neq 0, k_{10} \in \mathbb{R}\}$,
$\{\langle k_{11}V_1 + V_2 - V_3, a_{16}V_2 + V_3, V_4 + V_5\rangle, a_{16} \neq 0, -1; k_{11} \neq 0\}$,
$\{\langle V_2 - V_3, a_{16}V_2 + V_3, V_1\rangle, a_{16} \neq -1\}$.

4. Reduction of PDEs

In this section, we reduce the two-phase mass flow system of PDEs into another systems of PDEs. The reduction procedure works for any infinitesimal symmetry and it goes as follows. The procedure assigns to an infinitesimal symmetry a characteristics equation, which is a system of ODEs. The solutions of the characteristics equation lead to new independent variables (similarity variables) and dependent variables (similarity forms). Substitution of these similarity variables and similarity forms in the given system of PDEs results in a reduced system.

The crucial fact discussed in Section 3 is that many infinitesimal symmetries will lead to similar solutions. For fundamentally different solutions emerging from infinitesimal symmetries, it is enough to use an optimal system of Lie subalgebras as it contains information about different types of solutions. In what follows, we restrict our analysis to the optimal system of one dimensional Lie subalgebras computed in Section 3.1 to reduce the system of PDEs (6)–(7). We present the reduced systems (although each of these systems can be computed easily with some tedious computation) because anyone who is further interested in studying these systems can directly solve and analyze one or more of these systems that lead to fundamentally different quantitative solutions. We note that it is also possible to use the optimal systems of two- and three-dimensional Lie subalgebras for the reduction but we do not discuss that in this paper.

(I) *Reduced PDEs associated to* $V_1 + kV_2 + V_5$: The characteristics equation for this case is

$$\frac{dx}{1} = \frac{dy}{k} = \frac{dt}{t} = \frac{dh_1}{-2h_1} = \frac{dh_2}{-2h_2} = \frac{du_1}{-u_1} = \frac{du_2}{-u_2}.$$

The similarity variables (w, t') are given by $w = y - kx$, $tt' = e^x$, and the similarity forms $(\hat{u}_i, \hat{h}_i)$ satisfy $u_i\hat{u}_i = e^{-x}$, $h_i\hat{h}_i = e^{-2x}$. This reduces the PDEs (6)–(7) to:

$$\frac{t'^2}{\hat{h}_1^2}\frac{\partial \hat{h}_1}{\partial t'} - \frac{t'}{(\hat{u}_1\hat{h}_1)^2}\frac{\partial(\hat{u}_1\hat{h}_1)}{\partial t'} + \frac{k}{(\hat{u}_1\hat{h}_1)^2}\frac{\partial(\hat{u}_1\hat{h}_1)}{\partial w} = \frac{3}{\hat{u}_1\hat{h}_1},$$
$$\frac{t'^2}{\hat{h}_2^2}\frac{\partial \hat{h}_2}{\partial t'} - \frac{1}{(\hat{u}_2\hat{h}_2)^2}\frac{\partial(\hat{u}_2\hat{h}_2)}{\partial w} = 0, \tag{15}$$

$$\left(\frac{t'^2}{\hat{u}_1^2}-\frac{t'}{\hat{u}_1^3}\right)\frac{\partial\hat{u}_1}{\partial t'}+\frac{k}{\hat{u}_1^3}\frac{\partial\hat{u}_1}{\partial w}-\left(\frac{t'\beta_1}{\hat{h}_1^2}+\frac{t'\beta_1}{2\hat{h}_1\hat{h}_2}\right)\frac{\partial\hat{h}_1}{\partial t'}+\left(\frac{k\beta_1}{\hat{h}_1^2}+\frac{k\beta_1}{2\hat{h}_1\hat{h}_2}\right)\frac{\partial\hat{h}_1}{\partial w}$$
$$-\frac{t'\beta_1}{2\hat{h}_2^2}\frac{\partial\hat{h}_2}{\partial t'}+\frac{k\beta_1}{2\hat{h}_2^2}\frac{\partial\hat{h}_2}{\partial w}=\frac{1}{\hat{u}_1^2}+\frac{2\beta_1}{\hat{h}_1}+\frac{2\beta_1}{\hat{h}_2},$$
$$\frac{t'^2}{\hat{u}_2^2}\frac{\partial\hat{u}_2}{\partial t'}-\frac{1}{\hat{u}_2^3}\frac{\partial\hat{u}_2}{\partial w}-\frac{\beta_2}{\hat{h}_2^2}\frac{\partial\hat{h}_2}{\partial w}-\frac{\beta_2}{2\hat{h}_1^2}\frac{\partial\hat{h}_1}{\partial w}-\frac{\beta_2}{2\hat{h}_1\hat{h}_2}\frac{\partial\hat{h}_2}{\partial w}=0. \quad (16)$$

There are some particularly interesting aspects associated with the system (15)–(16). First, although this system has only one spatial variable w that combines (together with k) the two variables x and y from the system (6)–(7), the new system turns non-homogeneous for both the mass and momentum balance equations. Also, the coefficients in the reduced PDEs are highly non-linear both in similarity variables and similarity forms. These aspects made the reduced set a very challenging system of equations. This discussion also holds for several other reduced systems that follows.

(II) *Reduced PDEs associated to* $-V_1+kV_2+V_5$:
Characteristics equation:

$$\frac{dx}{-1}=\frac{dy}{k}=\frac{dt}{t}=\frac{dh_1}{-2h_1}=\frac{dh_2}{-2h_2}=\frac{du_1}{-u_1}=\frac{du_2}{-u_2}.$$

Similarity variables:

$w,\ t'$ with $w=y+kx,\ tt'=e^{-x}$.

Similarity forms:

$\hat{u}_i,\ \hat{h}_i$ with $u_i\hat{u}_i=e^{x},\ h_i\hat{h}_i=e^{2x}$.

Reduced PDEs:

$$\frac{t'^2}{\hat{h}_1^2}\frac{\partial\hat{h}_1}{\partial t'}+\frac{t'}{(\hat{u}_1\hat{h}_1)^2}\frac{\partial(\hat{u}_1\hat{h}_1)}{\partial t'}-\frac{k}{(\hat{u}_1\hat{h}_1)^2}\frac{\partial(\hat{u}_1\hat{h}_1)}{\partial w}=\frac{-3}{\hat{u}_1\hat{h}_1},$$
$$\frac{t'^2}{\hat{h}_2^2}\frac{\partial\hat{h}_2}{\partial t'}-\frac{1}{(\hat{u}_2\hat{h}_2)^2}\frac{\partial(\hat{u}_2\hat{h}_2)}{\partial w}=0, \quad (17)$$

$$\left(\frac{t'^2}{\hat{u}_1^2}+\frac{t'}{\hat{u}_1^3}\right)\frac{\partial\hat{u}_1}{\partial t'}-\frac{k}{\hat{u}_1^3}\frac{\partial\hat{u}_1}{\partial w}+\left(\frac{t'\beta_1}{\hat{h}_1^2}+\frac{t'\beta_1}{2\hat{h}_1\hat{h}_2}\right)\frac{\partial\hat{h}_1}{\partial t'}-\left(\frac{k\beta_1}{\hat{h}_1^2}+\frac{k\beta_1}{2\hat{h}_1\hat{h}_2}\right)\frac{\partial\hat{h}_1}{\partial w}$$
$$+\frac{t'\beta_1}{2\hat{h}_2^2}\frac{\partial\hat{h}_2}{\partial t'}-\frac{k\beta_1}{2\hat{h}_2^2}\frac{\partial\hat{h}_2}{\partial w}=-\frac{1}{\hat{u}_1^2}-\frac{2\beta_1}{\hat{h}_1}-\frac{2\beta_1}{\hat{h}_2},$$
$$\frac{t'^2}{\hat{u}_2^2}\frac{\partial\hat{u}_2}{\partial t'}-\frac{1}{\hat{u}_2^3}\frac{\partial\hat{u}_2}{\partial w}-\frac{\beta_2}{\hat{h}_2^2}\frac{\partial\hat{h}_2}{\partial w}-\frac{\beta_2}{2\hat{h}_1^2}\frac{\partial\hat{h}_1}{\partial w}-\frac{\beta_2}{2\hat{h}_1\hat{h}_2}\frac{\partial\hat{h}_2}{\partial w}=0. \quad (18)$$

(III) *Reduced PDEs associated to* V_2+V_5:
Characteristics equation:

$$\frac{dy}{1}=\frac{dt}{t}=\frac{dh_1}{-2h_1}=\frac{dh_2}{-2h_2}=\frac{du_1}{-u_1}=\frac{du_2}{-u_2}.$$

Similarity variables:

$x,\ t'$ with $tt'=e^{y}$.

Similarity forms:

$\hat{u}_i,\ \hat{h}_i$ with $u_i\hat{u}_i=e^{-y},\ h_i\hat{h}_i=e^{-2y}$.

Reduced PDEs:

$$\frac{t'^2}{\hat{h}_1^2}\frac{\partial\hat{h}_1}{\partial t'}-\frac{1}{(\hat{u}_1\hat{h}_1)^2}\frac{\partial(\hat{u}_1\hat{h}_1)}{\partial x}=0,\ \frac{t'^2}{\hat{h}_2^2}\frac{\partial\hat{h}_2}{\partial t'}-\frac{t'}{(\hat{u}_2\hat{h}_2)^2}\frac{\partial(\hat{u}_2\hat{h}_2)}{\partial t'}=\frac{3}{\hat{u}_2\hat{h}_2}, \quad (19)$$

$$\frac{t'^2}{\hat{u}_1^2}\frac{\partial\hat{u}_1}{\partial t'}-\frac{1}{\hat{u}_1^3}\frac{\partial\hat{u}_1}{\partial x}-\frac{\beta_1}{\hat{h}_1^2}\frac{\partial\hat{h}_1}{\partial x}-\frac{\beta_1}{2\hat{h}_2^2}\frac{\partial\hat{h}_2}{\partial x}-\frac{\beta_1}{2\hat{h}_1\hat{h}_2}\frac{\partial\hat{h}_1}{\partial x}=0,$$
$$\left(\frac{t'^2}{\hat{u}_2^2}-\frac{t'}{\hat{u}_2^3}\right)\frac{\partial\hat{u}_2}{\partial t'}-\frac{\beta_2t'}{\hat{h}_2^2}\frac{\partial\hat{h}_2}{\partial t'}-\frac{\beta_2t'}{2\hat{h}_1^2}\frac{\partial\hat{h}_1}{\partial t'}-\frac{\beta_2t'}{2\hat{h}_1\hat{h}_2}\frac{\partial\hat{h}_2}{\partial t'}=\frac{1}{\hat{u}_2^2}+\frac{2\beta_2}{\hat{h}_2}+\frac{2\beta_2}{\hat{h}_1}. \quad (20)$$

(IV) *Reduced PDEs associated to* $-V_2+V_5$:
Characteristics equation:

$$\frac{dy}{-1}=\frac{dt}{t}=\frac{dh_1}{-2h_1}=\frac{dh_2}{-2h_2}=\frac{du_1}{-u_1}=\frac{du_2}{-u_2}.$$

Similarity variables:

$x,\ t'$ with $tt'=e^{-y}$.

Similarity forms:

$\hat{u}_i,\ \hat{h}_i$ with $u_i\hat{u}_i=e^{y},\ h_i\hat{h}_i=e^{2y}$.

Reduced PDEs:

$$\frac{t'^2}{\hat{h}_1^2}\frac{\partial\hat{h}_1}{\partial t'}-\frac{1}{(\hat{u}_1\hat{h}_1)^2}\frac{\partial(\hat{u}_1\hat{h}_1)}{\partial x}=0,\ \frac{t'^2}{\hat{h}_2^2}\frac{\partial\hat{h}_2}{\partial t'}+\frac{t'}{(\hat{u}_2\hat{h}_2)^2}\frac{\partial(\hat{u}_2\hat{h}_2)}{\partial t'}=\frac{-3}{\hat{u}_2\hat{h}_2}, \quad (21)$$

$$\frac{t'^2}{\hat{u}_1^2}\frac{\partial\hat{u}_1}{\partial t'}-\frac{1}{\hat{u}_1^3}\frac{\partial\hat{u}_1}{\partial x}-\frac{\beta_1}{\hat{h}_1^2}\frac{\partial\hat{h}_1}{\partial x}-\frac{\beta_1}{2\hat{h}_2^2}\frac{\partial\hat{h}_2}{\partial x}-\frac{\beta_1}{2\hat{h}_1\hat{h}_2}\frac{\partial\hat{h}_1}{\partial x}=0,$$
$$\left(\frac{t'^2}{\hat{u}_2^2}+\frac{t'}{\hat{u}_2^3}\right)\frac{\partial\hat{u}_2}{\partial t'}+\frac{\beta_2t'}{\hat{h}_2^2}\frac{\partial\hat{h}_2}{\partial t'}+\frac{\beta_2t'}{2\hat{h}_1^2}\frac{\partial\hat{h}_1}{\partial t'}+\frac{\beta_2t'}{2\hat{h}_1\hat{h}_2}\frac{\partial\hat{h}_2}{\partial t'}$$
$$=-\frac{1}{\hat{u}_2^2}-\frac{2\beta_2}{\hat{h}_2}-\frac{2\beta_2}{\hat{h}_1}. \quad (22)$$

(V) *Reduced PDEs associated to* V_5:
Characteristics equation:

$$\frac{dt}{t}=\frac{-dh_1}{2h_1}=\frac{-dh_2}{2h_2}=\frac{-du_1}{u_1}=\frac{-du_2}{u_2}.$$

Similarity forms:

$\hat{u}_i,\ \hat{h}_i$ with $\hat{u}_i=tu_i,\ \hat{h}_i=t^2h_i$.

Reduced PDEs:

$$t\frac{\partial\hat{h}_1}{\partial t}+\frac{\partial(\hat{u}_1\hat{h}_1)}{\partial x}=2\hat{h}_1,\ t\frac{\partial\hat{h}_2}{\partial t}+\frac{\partial(\hat{u}_2\hat{h}_2)}{\partial x}=2\hat{h}_2, \quad (23)$$

$$t\frac{\partial\hat{u}_1}{\partial t}+\hat{u}_1\frac{\partial\hat{u}_1}{\partial x}+\beta_1\frac{\partial\hat{h}_1}{\partial x}+\frac{\beta_1}{2}\frac{\partial\hat{h}_2}{\partial x}+\frac{\beta_1}{2}\frac{\hat{h}_2}{\hat{h}_1}\frac{\partial\hat{h}_1}{\partial x}=\hat{u}_1,$$
$$t\frac{\partial\hat{u}_2}{\partial t}+\hat{u}_2\frac{\partial\hat{u}_2}{\partial y}+\beta_2\frac{\partial\hat{h}_2}{\partial y}+\frac{\beta_2}{2}\frac{\partial\hat{h}_1}{\partial y}+\frac{\beta_2}{2}\frac{\hat{h}_1}{\hat{h}_2}\frac{\partial\hat{h}_2}{\partial y}=\hat{u}_2. \quad (24)$$

(VI) *Reduced PDEs with the symmetry* V_3+V_4:
Characteristics equation:

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dt}{1}=\frac{dh_1}{2h_1}=\frac{dh_2}{2h_2}=\frac{du_1}{u_1}=\frac{du_2}{u_2}.$$

Similarity variables:

$w,\ t'$ with $y=wx,\ \ t'x=e^{t}$.

Similarity forms:

$\hat{u}_i,\ \hat{h}_i$ with $x\hat{u}_i=u_i,\ \ x^2\hat{h}_i=h_i$.

Reduced PDEs:

$$t'\frac{\partial\hat{h}_1}{\partial t'}-w\frac{\partial(\hat{u}_1\hat{h}_1)}{\partial w}-t'\frac{\partial(\hat{u}_1\hat{h}_1)}{\partial t'}=-3\hat{u}_1\hat{h}_1,\ t'\frac{\partial\hat{h}_2}{\partial t'}+\frac{\partial(\hat{u}_2\hat{h}_2)}{\partial w}=0, \quad (25)$$

$$t'\frac{\partial\hat{u}_1}{\partial t'}-\hat{u}_1w\frac{\partial\hat{u}_1}{\partial w}-\beta_1w\frac{\partial\hat{h}_1}{\partial w}-\frac{\beta_1}{2}w\frac{\partial\hat{h}_2}{\partial w}-\frac{\beta_1}{2}w\frac{\hat{h}_2}{\hat{h}_1}\frac{\partial\hat{h}_1}{\partial w}-t'\hat{u}_1\frac{\partial\hat{u}_1}{\partial t'}$$
$$-\beta_1t'\frac{\partial\hat{h}_1}{\partial t'}-\frac{\beta_1}{2}t'\frac{\partial\hat{h}_2}{\partial t'}-\frac{\beta_1}{2}t'\frac{\hat{h}_2}{\hat{h}_1}\frac{\partial\hat{h}_1}{\partial t'}=-\hat{u}_1^2-2\beta_1\hat{h}_1-2\beta_1\hat{h}_2,$$
$$t'\frac{\partial\hat{u}_2}{\partial t'}+\hat{u}_2\frac{\partial\hat{u}_2}{\partial w}+\beta_2\frac{\partial\hat{h}_2}{\partial w}+\frac{\beta_2}{2}\frac{\partial\hat{h}_1}{\partial w}+\frac{\beta_2}{2}\frac{\hat{h}_1}{\hat{h}_2}\frac{\partial\hat{h}_2}{\partial w}=0. \quad (26)$$

Similarly, we can deal with $-V_3+V_4$.

(VII) *Reduced PDEs with the symmetry* V_4+kV_5:
Characteristics equation:

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dt}{kt}=\frac{dh_1}{2h_1(1-k)}=\frac{dh_2}{2h_2(1-k)}=\frac{du_1}{u_1(1-k)}=\frac{du_2}{u_2(1-k)}.$$

Similarity variables:

$w,\ t'$ with $y=wx,\ t'x^k=t$.

Similarity forms:

$\hat{u}_i,\ \hat{h}_i$ with $x^{(1-k)}\hat{u}_i=u_i,\ x^{2(1-k)}\hat{h}_i=h_i$.

Reduced PDEs:

$$\frac{\partial\hat{h}_1}{\partial t'}-w\frac{\partial(\hat{u}_1\hat{h}_1)}{\partial w}-kt'\frac{\partial(\hat{u}_1\hat{h}_1)}{\partial t'}=-3(1-k)\hat{u}_1\hat{h}_1,\ \frac{\partial\hat{h}_2}{\partial t'}+\frac{\partial(\hat{u}_2\hat{h}_2)}{\partial w}=0, \tag{27}$$

$$\begin{aligned}&\frac{\partial\hat{u}_1}{\partial t'}-\hat{u}_1w\frac{\partial\hat{u}_1}{\partial w}-\beta_1w\frac{\partial\hat{h}_1}{\partial w}-\frac{\beta_1}{2}w\frac{\partial\hat{h}_2}{\partial w}-\frac{\beta_1}{2}w\frac{\hat{h}_2}{\hat{h}_1}\frac{\partial\hat{h}_1}{\partial w}-\hat{u}_1kt'\frac{\partial\hat{u}_1}{\partial t'}-\beta_1kt'\frac{\partial\hat{h}_1}{\partial t'}\\&\quad-\frac{\beta_1}{2}kt'\frac{\partial\hat{h}_2}{\partial t'}-\frac{\beta_1}{2}kt'\frac{\hat{h}_2}{\hat{h}_1}\frac{\partial\hat{h}_1}{\partial t'}=(1-k)\{-\hat{u}_1^2-2\beta_1\hat{h}_1-2\beta_1\hat{h}_2\},\\&\frac{\partial\hat{u}_2}{\partial t'}+\hat{u}_2\frac{\partial\hat{u}_2}{\partial w}+\beta_2\frac{\partial\hat{h}_2}{\partial w}+\frac{\beta_2}{2}\frac{\partial\hat{h}_1}{\partial w}+\frac{\beta_2}{2}\frac{\hat{h}_1}{\hat{h}_2}\frac{\partial\hat{h}_2}{\partial w}=0.\end{aligned} \tag{28}$$

We observe that this system reduces to a homogeneous system for k=1, it is expected as $V_4+V_5=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+t\frac{\partial}{\partial t}$ is a scaling transformation in the independent variables.

(VIII) *Reduced PDEs with the symmetry* $kV_1+V_2+V_3$:

Characteristics equation:

$$\frac{dx}{k}=\frac{dy}{1}=\frac{dt}{1}.$$

Similarity variables:

$w,\ t'$ with $w=y-\frac{x}{k},\ t'=t-\frac{x}{k}$.

Similarity forms:

$\hat{u}_i,\ \hat{h}_i$ with $\hat{u}_i=u_i,\ \hat{h}_i=h_i$.

Reduced PDEs:

$$\frac{\partial\hat{h}_1}{\partial t'}-\frac{1}{k}\frac{\partial(\hat{u}_1\hat{h}_1)}{\partial w}-\frac{1}{k}\frac{\partial(\hat{u}_1\hat{h}_1)}{\partial t'}=0,\ \frac{\partial\hat{h}_2}{\partial t'}+\frac{\partial(\hat{u}_2\hat{h}_2)}{\partial w}=0, \tag{29}$$

$$\begin{aligned}&\frac{\partial\hat{u}_1}{\partial t'}-\frac{1}{k}\hat{u}_1\frac{\partial\hat{u}_1}{\partial w}-\frac{1}{k}\beta_1\frac{\partial\hat{h}_1}{\partial w}-\frac{\beta_1}{2k}\frac{\partial\hat{h}_2}{\partial w}-\frac{\beta_1}{2k}\frac{\hat{h}_2}{\hat{h}_1}\frac{\partial\hat{h}_1}{\partial w}-\frac{1}{k}\hat{u}_1\frac{\partial\hat{u}_1}{\partial t'}-\frac{1}{k}\beta_1\frac{\partial\hat{h}_1}{\partial t'}\\&\quad-\frac{\beta_1}{2k}\frac{\partial\hat{h}_2}{\partial t'}-\frac{\beta_1}{2k}\frac{\hat{h}_2}{\hat{h}_1}\frac{\partial\hat{h}_1}{\partial t'}=0,\\&\frac{\partial\hat{u}_2}{\partial t'}+\hat{u}_2\frac{\partial\hat{u}_2}{\partial w}+\beta_2\frac{\partial\hat{h}_2}{\partial w}+\frac{\beta_2}{2}\frac{\partial\hat{h}_1}{\partial w}+\frac{\beta_2}{2}\frac{\hat{h}_1}{\hat{h}_2}\frac{\partial\hat{h}_2}{\partial w}=0.\end{aligned} \tag{30}$$

As $kV_1+V_2+V_3$ is a linear combination of translations in the independent variables, the reduced system is a homogeneous system as expected. Similarly, we can deal with $kV_1+V_2-V_3$, $kV_1-V_2+V_3$, $kV_1-V_2-V_3$.

(IX) *Reduced PDEs with the symmetry* V_1+V_3:

Characteristics equation:

$$\frac{dx}{1}=\frac{dt}{1}.$$

Similarity variables:

$y,\ t'$ with $t'=t-x$.

Similarity forms:

$\hat{u}_i,\ \hat{h}_i$ with $\hat{u}_i=u_i,\ \hat{h}_i=h_i$.

Reduced PDEs:

$$\frac{\partial\hat{h}_1}{\partial t'}-\frac{\partial(\hat{u}_1\hat{h}_1)}{\partial t'}=0,\ \frac{\partial\hat{h}_2}{\partial t'}-\frac{\partial(\hat{u}_2\hat{h}_2)}{\partial t'}=0, \tag{31}$$

$$\begin{aligned}&\frac{\partial\hat{u}_1}{\partial t'}-\hat{u}_1\frac{\partial\hat{u}_1}{\partial t'}-\beta_1\frac{\partial\hat{h}_1}{\partial t'}-\frac{\beta_1}{2}\frac{\partial\hat{h}_2}{\partial t'}-\frac{\beta_1}{2}\frac{\hat{h}_2}{\hat{h}_1}\frac{\partial\hat{h}_1}{\partial t'}=0,\\&\frac{\partial\hat{u}_2}{\partial t'}+\hat{u}_2\frac{\partial\hat{u}_2}{\partial y}+\beta_2\frac{\partial\hat{h}_2}{\partial y}+\frac{\beta_2}{2}\frac{\partial\hat{h}_1}{\partial y}+\frac{\beta_2}{2}\frac{\hat{h}_1}{\hat{h}_2}\frac{\partial\hat{h}_2}{\partial y}=0.\end{aligned} \tag{32}$$

Similarly, we can compute the reduced PDEs for the symmetries V_1-V_3, V_2+V_3, V_2-V_3 and kV_1+V_2 and each of these symmetries yields a homogeneous system. These cases are similar to the one discussed in [1].

5. Further reduction

Here, we discuss the further reduction of some of the reduced system of PDEs from Section 4. There are at least two ways to deal with this problem. One way, the challenging one is to compute all the possible symmetries of each of the reduced systems. Instead, we follow a different approach, where we compute some symmetries from the knowledge of the Lie brackets and use them for further reduction. Some of the reduced systems are ODEs and some are still PDEs.

The Lie bracket, which is an abstract mathematical object, surprisingly contains information about whether a second reduction is possible. More precisely, if V and W are two infinitesimal symmetries such that $[V,W]=\lambda V$ for some constant λ, then, W induces a symmetry of the reduced system of PDEs obtained by using V [11,29]. Following this fact, we compute symmetries of some reduced systems of PDEs from Section 4 and use them for reduction.

Recall from Section 2

$$[V_1+kV_2+V_5,\ V_2]=0\quad\text{and}\quad[V_1+kV_2+V_5,\ V_5]=0.$$

Thus, V_2 and V_5 induce infinitesimal symmetries of the reduced system (15)–(16). The infinitesimal symmetries obtained in this way are

$$\frac{\partial}{\partial w}\quad\text{and}\quad t'\frac{\partial}{\partial t'}-2\hat{h}_1\frac{\partial}{\partial\hat{h}_1}-2\hat{h}_2\frac{\partial}{\partial\hat{h}_2}-\hat{u}_1\frac{\partial}{\partial\hat{u}_1}-\hat{u}_2\frac{\partial}{\partial\hat{u}_2}.$$

We can also check directly that these are symmetries of (15)–(16).

(I) *ODEs associated with* (15)–(16): The discussion in the previous paragraph implies (15)–(16) has a two dimensional Lie algebra of infinitesimal symmetries:

$$b_1\frac{\partial}{\partial w}+b_2\left(t'\frac{\partial}{\partial t'}-2\hat{h}_1\frac{\partial}{\partial\hat{h}_1}-2\hat{h}_2\frac{\partial}{\partial\hat{h}_2}-\hat{u}_1\frac{\partial}{\partial\hat{u}_1}-\hat{u}_2\frac{\partial}{\partial\hat{u}_2}\right) \tag{33}$$

where b_1 and b_2 are real constants.

Characteristics equation:

$$\frac{dw}{b_1}=\frac{dt'}{b_2t'}=\frac{d\hat{h}_1}{-2b_2\hat{h}_1}=\frac{d\hat{h}_2}{-2b_2\hat{h}_2}=\frac{d\hat{u}_1}{-b_2\hat{u}_1}=\frac{d\hat{u}_2}{-b_2\hat{u}_2}.$$

In what follows, we fix $l=b_2/b_1$. For the infinitesimal symmetry (33), we can compute

Similarity variable:

v with $e^{lw}=vt'$,

Similarity forms:

$\tilde{u}_i,\ \tilde{h}_i$ with $\hat{h}_i\tilde{h}_i=e^{-2lw}$, $\quad\hat{u}_i\tilde{u}_i=e^{-lw}$ for $i=1,2$.

Substitution of these similarity variables and similarity forms in (15)–(16) results in the following system of ODEs:

$$\begin{bmatrix} 1-(kl+1)v\tilde{u}_1 & 0 & -(kl+1)v\tilde{h}_1 & 0 \\ 0 & 1+lv\tilde{u}_2 & 0 & lv\tilde{h}_2 \\ -\beta_1 v(1+kl)(1+0.5\tilde{h}_2/\tilde{h}_1) & -0.5\beta_1 v(1+kl) & 1-v(1+kl)\tilde{u}_1 & 0 \\ 0.5l\beta_2 v & \beta_2 lv(1+0.5\tilde{h}_1/\tilde{h}_2) & 0 & (1+lv\tilde{u}_2) \end{bmatrix} \begin{bmatrix} d\tilde{h}_1/dv \\ d\tilde{h}_2/dv \\ d\tilde{u}_1/dv \\ d\tilde{u}_2/dv \end{bmatrix} = \begin{bmatrix} 3(1+kl)\tilde{u}_1\tilde{h}_1 \\ -3l\tilde{u}_2\tilde{h}_2 \\ (1+kl)\tilde{u}_1^2+2(1+kl)\beta_1\tilde{h}_1+2(1+kl)\beta_1\tilde{h}_2 \\ -l(\tilde{u}_2^2+2\beta_2\tilde{h}_1+2\beta_2\tilde{h}_2) \end{bmatrix}. \tag{34}$$

Similar analysis can be done for the other systems of PDEs in Section 4. We will only write the outcomes below.

(II) *ODEs associated with* (17)–(18):

Lie algebra of infinitesimal symmetries:

$$b_1\frac{\partial}{\partial w}+b_2\left(t'\frac{\partial}{\partial t'}-2\hat{h}_1\frac{\partial}{\partial \hat{h}_1}-2\hat{h}_2\frac{\partial}{\partial \hat{h}_2}-\hat{u}_1\frac{\partial}{\partial \hat{u}_1}-\hat{u}_2\frac{\partial}{\partial \hat{u}_2}\right)$$

Characteristics equation:

$$\frac{dw}{b_1}=\frac{dt'}{b_2t'}=\frac{d\hat{h}_1}{-2b_2\hat{h}_1}=\frac{d\hat{h}_2}{-2b_2\hat{h}_2}=\frac{d\hat{u}_1}{-b_2\hat{u}_1}=\frac{d\hat{u}_2}{-b_2\hat{u}_2}.$$

Similarity variable:

v where $e^{lw}=vt'$.

Similarity forms:

$\tilde{u}_i,\ \tilde{h}_i$ with $\hat{h}_i\tilde{h}_i=e^{-2lw},\quad \hat{u}_i\tilde{u}_i=e^{-lw}$ for $i=1,2$.

Reduced system of ODEs:

$$\begin{bmatrix} 1+(kl+1)v\tilde{u}_1 & 0 & (kl+1)v\tilde{h}_1 & 0 \\ 0 & 1+lv\tilde{u}_2 & 0 & lv\tilde{h}_2 \\ \beta_1 v(1+kl)(1+0.5\tilde{h}_2/\tilde{h}_1) & 0.5\beta_1 v(1+kl) & 1+v(1+kl)\tilde{u}_1 & 0 \\ 0.5l\beta_2 v & \beta_2 lv(1+0.5\tilde{h}_1/\tilde{h}_2) & 0 & (1+lv\tilde{u}_2) \end{bmatrix} \begin{bmatrix} d\tilde{h}_1/dv \\ d\tilde{h}_2/dv \\ d\tilde{u}_1/dv \\ d\tilde{u}_2/dv \end{bmatrix} = \begin{bmatrix} -3(1+kl)\tilde{u}_1\tilde{h}_1 \\ -3l\tilde{u}_2\tilde{h}_2 \\ -(1+kl)\tilde{u}_1^2-2(1+kl)\beta_1\tilde{h}_1-2(1+kl)\beta_1\tilde{h}_2 \\ -l(\tilde{u}_2^2+2\beta_2\tilde{h}_1+2\beta_2\tilde{h}_2) \end{bmatrix}. \tag{35}$$

(III) *ODEs associated with* (19)–(20):

Lie algebra of infinitesimal symmetries:

$$b_1\frac{\partial}{\partial x}+b_2\left(t'\frac{\partial}{\partial t'}-2\hat{h}_1\frac{\partial}{\partial \hat{h}_1}-2\hat{h}_2\frac{\partial}{\partial \hat{h}_2}-\hat{u}_1\frac{\partial}{\partial \hat{u}_1}-\hat{u}_2\frac{\partial}{\partial \hat{u}_2}\right)$$

Characteristics equation:

$$\frac{dx}{b_1}=\frac{dt'}{b_2t'}=\frac{d\hat{h}_1}{-2b_2\hat{h}_1}=\frac{d\hat{h}_2}{-2b_2\hat{h}_2}=\frac{d\hat{u}_1}{-b_2\hat{u}_1}=\frac{d\hat{u}_2}{-b_2\hat{u}_2}.$$

Similarity variable:

v where $e^{lx}=vt'$.

Similarity forms:

$\tilde{u}_i,\ \tilde{h}_i$ with $\hat{h}_i\tilde{h}_i=e^{-2lx},\quad \hat{u}_i\tilde{u}_i=e^{-lx}$ for $i=1,2$.

Reduced system of ODEs:

$$\begin{bmatrix} 1+lv\tilde{u}_1 & 0 & lv\tilde{h}_1 & 0 \\ 0 & 1-v\tilde{u}_2 & 0 & -v\tilde{h}_2 \\ \beta_1 lv+0.5\beta_1 lv\tilde{h}_2/\tilde{h}_1 & 0.5\beta_1 lv & 1+lv\tilde{u}_1 & 0 \\ -0.5\beta_2 v & -\beta_2 v-0.5\beta_2 v\tilde{h}_1/\tilde{h}_2 & 0 & 1-v\tilde{u}_2 \end{bmatrix} \begin{bmatrix} d\tilde{h}_1/dv \\ d\tilde{h}_2/dv \\ d\tilde{u}_1/dv \\ d\tilde{u}_2/dv \end{bmatrix} = \begin{bmatrix} -3l\tilde{u}_1\tilde{h}_1 \\ 3\tilde{u}_2\tilde{h}_2 \\ -l(\tilde{u}_1^2+2\beta_1\tilde{h}_1+2\beta_1\tilde{h}_2) \\ \tilde{u}_2^2+2\beta_2\tilde{h}_1+2\beta_2\tilde{h}_2 \end{bmatrix}. \tag{36}$$

(IV) *ODEs associated with* (21)–(22):

A Lie algebra of infinitesimal symmetries:

$$b_1\frac{\partial}{\partial x}+b_2\left(t'\frac{\partial}{\partial t'}-2\hat{h}_1\frac{\partial}{\partial \hat{h}_1}-2\hat{h}_2\frac{\partial}{\partial \hat{h}_2}-\hat{u}_1\frac{\partial}{\partial \hat{u}_1}-\hat{u}_2\frac{\partial}{\partial \hat{u}_2}\right)$$

Characteristics equation:

$$\frac{dx}{b_1}=\frac{dt'}{b_2t'}=\frac{d\hat{h}_1}{-2b_2\hat{h}_1}=\frac{d\hat{h}_2}{-2b_2\hat{h}_2}=\frac{d\hat{u}_1}{-b_2\hat{u}_1}=\frac{d\hat{u}_2}{-b_2\hat{u}_2}.$$

Similarity variable:

v where $e^{lx}=vt'$.

Similarity forms:

$\tilde{u}_i,\ \tilde{h}_i$ with $\hat{h}_i\tilde{h}_i=e^{-2lx},\quad \hat{u}_i\tilde{u}_i=e^{-lx}$ for $i=1,2$.

Reduced system of ODEs:

$$\begin{bmatrix} 1+lv\tilde{u}_1 & 0 & lv\tilde{h}_1 & 0 \\ 0 & 1+v\tilde{u}_2 & 0 & v\tilde{h}_2 \\ \beta_1 lv+0.5\beta_1 lv\tilde{h}_2/\tilde{h}_1 & 0.5\beta_1 lv & 1+lv\tilde{u}_1 & 0 \\ 0.5\beta_2 v & \beta_2 v+0.5\beta_2 v\tilde{h}_1/\tilde{h}_2 & 0 & 1+v\tilde{u}_2 \end{bmatrix} \begin{bmatrix} d\tilde{h}_1/dv \\ d\tilde{h}_2/dv \\ d\tilde{u}_1/dv \\ d\tilde{u}_2/dv \end{bmatrix} = \begin{bmatrix} -3l\tilde{u}_1\tilde{h}_1 \\ -3\tilde{u}_2\tilde{h}_2 \\ -l(\tilde{u}_1^2+2\beta_1\tilde{h}_1+2\beta_1\tilde{h}_2) \\ -(\tilde{u}_2^2+2\beta_2\tilde{h}_1+2\beta_2\tilde{h}_2) \end{bmatrix}. \tag{37}$$

(V) *PDEs associated with* (23)–(24):

A Lie algebra of infitesimal symmetries:

$$b_1\frac{\partial}{\partial x}+b_2\frac{\partial}{\partial y}+b_3\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+2\hat{h}_1\frac{\partial}{\partial \hat{h}_1}+2\hat{h}_2\frac{\partial}{\partial \hat{h}_2}+\hat{u}_1\frac{\partial}{\partial \hat{u}_1}+\hat{u}_2\frac{\partial}{\partial \hat{u}_2}\right)$$

Characteristics equation:

$$\frac{dx}{b_1+b_3x}=\frac{dy}{b_2+b_3y}=\frac{d\hat{h}_1}{2b_3\hat{h}_1}=\frac{d\hat{h}_2}{2b_3\hat{h}_2}=\frac{d\hat{u}_1}{b_3\hat{u}_1}=\frac{d\hat{u}_2}{b_3\hat{u}_2}.$$

Similarity variable:

$w,\ t$ with $w(b_1+b_3x)=(b_2+b_3y)$.

Similarity forms:

$\tilde{u}_i,\ \tilde{h}_i$ with $\hat{h}_i=(b_1+b_3x)^2\tilde{h}_i,\quad \hat{u}_i=(b_1+b_3x)\tilde{u}_i$ for $i=1,2$.

Reduced system of PDEs:

$$t\frac{\partial\tilde{h}_1}{\partial t}-wb_3\frac{\partial(\tilde{u}_1\tilde{h}_1)}{\partial w}=2\tilde{h}_1-3b_3\tilde{u}_1\tilde{h}_1, \tag{38}$$

$$t\frac{\partial\tilde{h}_2}{\partial t}+b_3\frac{\partial(\tilde{u}_2\tilde{h}_2)}{\partial w}=2\tilde{h}_2,$$

$$t\frac{\partial\tilde{u}_1}{\partial t} - b_3 w\tilde{u}_1\frac{\partial\tilde{u}_1}{\partial w} - \beta_1 b_3 w\frac{\partial\tilde{h}_1}{\partial w} - \frac{\beta_1 w}{2}b_3\frac{\partial\tilde{h}_2}{\partial w} - \frac{\beta_1 w}{2}b_3\frac{\tilde{h}_2}{\tilde{h}_1}\frac{\partial\tilde{h}_1}{\partial w} = -b_3\tilde{u}_1^2 - 2\beta_1 b_3\tilde{h}_1 - 2\beta_1 b_3\tilde{h}_2 + \tilde{u}_1,$$
$$t\frac{\partial\tilde{u}_2}{\partial t} + \tilde{u}_2 b_3\frac{\partial\tilde{u}_2}{\partial w} + \beta_2 b_3\frac{\partial\tilde{h}_2}{\partial w} + \frac{\beta_2}{2}b_3\frac{\partial\tilde{h}_1}{\partial w} + \frac{\beta_2}{2}\frac{\tilde{h}_1}{\tilde{h}_2}b_3\frac{\partial\tilde{h}_2}{\partial w} = \tilde{u}_2. \quad (39)$$

(VI) *PDEs associated with* (25)–(26):

A Lie algebra of infitesimal symmetries:

$$b_1 t'\frac{\partial}{\partial t'} + b_2\left(2\hat{h}_1\frac{\partial}{\partial\hat{h}_1} + 2\hat{h}_2\frac{\partial}{\partial\hat{h}_2} + \hat{u}_1\frac{\partial}{\partial\hat{u}_1} + \hat{u}_2\frac{\partial}{\partial\hat{u}_2}\right)$$

Characteristics equation:

$$\frac{dt'}{b_1 t'} = \frac{d\hat{h}_1}{2b_2\hat{h}_1} = \frac{d\hat{h}_2}{2b_2\hat{h}_2} = \frac{d\hat{u}_1}{b_2\hat{u}_1} = \frac{d\hat{u}_2}{b_2\hat{u}_2}.$$

Similarity forms:

$\tilde{u}_i, \tilde{h}_i$ with $\hat{h}_i = (t')^{2l}\tilde{h}_i, \quad \hat{u}_i = (t')^{l}\tilde{u}_i$ for $i = 1, 2.$

Reduced system of PDEs:

$$t'\frac{\partial\tilde{h}_1}{\partial t'} - wt'^{l}\frac{\partial(\tilde{u}_1\tilde{h}_1)}{\partial w} - t'^{(l+1)}\frac{\partial(\tilde{u}_1\tilde{h}_1)}{\partial t'} = -3t'^{l}\tilde{u}_1\tilde{h}_1(1-l) - 2l\tilde{h}_1,$$
$$t'\frac{\partial\tilde{h}_2}{\partial t'} - t'^{l}\frac{\partial(\tilde{u}_2\tilde{h}_2)}{\partial w} = -2l\tilde{h}_2, \quad (40)$$

$$t'\frac{\partial\tilde{u}_1}{\partial t'} - t'^{l}w\tilde{u}_1\frac{\partial\tilde{u}_1}{\partial w} - \beta_1 t'^{l}w\frac{\partial\tilde{h}_1}{\partial w} - \frac{\beta_1 w}{2}t'^{l}\frac{\partial\tilde{h}_2}{\partial w} - \frac{\beta_1 w}{2}t'^{l}\frac{\tilde{h}_2}{\tilde{h}_1}\frac{\partial\tilde{h}_1}{\partial w} - t'^{l+1}\tilde{u}_1\frac{\partial\tilde{u}_1}{\partial t'} - \beta_1 t'^{l+1}\frac{\partial\tilde{h}_1}{\partial t'} - \frac{\beta_1 t'^{l+1}}{2}\frac{\partial\tilde{h}_2}{\partial t'} - \frac{\beta_1}{2}t'^{l+1}\frac{\tilde{h}_2}{\tilde{h}_1}\frac{\partial\tilde{h}_1}{\partial t'} = (l-1)(t'^{l}\tilde{u}_1^2 + 2\beta_1 t'^{l}\tilde{h}_1 + 2\beta_1 t'^{l}\tilde{h}_2) - l\tilde{u}_1,$$
$$t'\frac{\partial\tilde{u}_2}{\partial t'} + \tilde{u}_2 t'^{l}\frac{\partial\tilde{u}_2}{\partial w} + \beta_2 t'^{l}\frac{\partial\tilde{h}_2}{\partial w} + \frac{\beta_2}{2}t'^{l}\frac{\partial\tilde{h}_1}{\partial w} + \frac{\beta_2}{2}\frac{\tilde{h}_1}{\tilde{h}_2}t'^{l}\frac{\partial\tilde{h}_2}{\partial w} = -l\tilde{u}_2. \quad (41)$$

(VII) *PDEs associated with* (27)–(28):

A Lie algebra of infitesimal symmetries:

$$b_1\left(2\hat{h}_1\frac{\partial}{\partial\hat{h}_1} + 2\hat{h}_2\frac{\partial}{\partial\hat{h}_2} + \hat{u}_1\frac{\partial}{\partial\hat{u}_1} + \hat{u}_2\frac{\partial}{\partial\hat{u}_2}\right) + b_2\left(t'\frac{\partial}{\partial t'} - 2\hat{h}_1\frac{\partial}{\partial\hat{h}_1} - 2\hat{h}_2\frac{\partial}{\partial\hat{h}_2} - \hat{u}_1\frac{\partial}{\partial\hat{u}_1} - \hat{u}_2\frac{\partial}{\partial\hat{u}_2}\right)$$

Characteristics equation:

$$\frac{dt'}{b_2 t'} = \frac{d\hat{h}_1}{2(b_1-b_2)\hat{h}_1} = \frac{d\hat{h}_2}{2(b_1-b_2)\hat{h}_2} = \frac{d\hat{u}_1}{(b_1-b_2)\hat{u}_1} = \frac{d\hat{u}_2}{(b_1-b_2)\hat{u}_2}.$$

Similarity forms:

$\tilde{u}_i, \tilde{h}_i$ with $\hat{h}_i = t'^{(2l-2)}\tilde{h}_i, \quad \hat{u}_i = t'^{(l-1)}\tilde{u}_i$ for $i = 1, 2.$

Reduced system of PDEs:

$$t'\frac{\partial\tilde{h}_1}{\partial t'} - wt'^{l}\frac{\partial(\tilde{u}_1\tilde{h}_1)}{\partial w} - kt'^{l+1}\frac{\partial(\tilde{u}_1\tilde{h}_1)}{\partial t'} = -2(l-1)\tilde{h}_1 + 3(kl-1)t'^{l}\tilde{u}_1\tilde{h}_1,$$
$$t'\frac{\partial\tilde{h}_2}{\partial t'} - t'^{l}\frac{\partial(\tilde{u}_2\tilde{h}_2)}{\partial w} = -(2l-2)\tilde{h}_2, \quad (42)$$

$$t'\frac{\partial\tilde{u}_1}{\partial t'} - t'^{l}w\tilde{u}_1\frac{\partial\tilde{u}_1}{\partial w} - \beta_1 t'^{l}w\frac{\partial\tilde{h}_1}{\partial w} - \frac{\beta_1 w}{2}t'^{l}\frac{\partial\tilde{h}_2}{\partial w} - \frac{\beta_1 w}{2}t'^{l}\frac{\tilde{h}_2}{\tilde{h}_1}\frac{\partial\tilde{h}_1}{\partial w} - t'^{l+1}k\tilde{u}_1\frac{\partial\tilde{u}_1}{\partial t'} - \beta_1 kt'^{l+1}\frac{\partial\tilde{h}_1}{\partial t'} - \frac{\beta_1 k}{2}t'^{l+1}\frac{\partial\tilde{h}_2}{\partial t'} - \frac{\beta_1 k}{2}t'^{l+1}\frac{\tilde{h}_2}{\tilde{h}_1}\frac{\partial\tilde{h}_1}{\partial t'} = (kl-1)t'^{l}(\tilde{u}_1^2 + 2\beta_1\tilde{h}_1 + 2\beta_1\tilde{h}_2) - (l-1)\tilde{u}_1,$$
$$t'\frac{\partial\tilde{u}_2}{\partial t'} + \tilde{u}_2 t'^{l}\frac{\partial\tilde{u}_2}{\partial w} + \beta_2 t'^{l}\frac{\partial\tilde{h}_2}{\partial w} + \frac{\beta_2}{2}t'^{l}\frac{\partial\tilde{h}_1}{\partial w} + \frac{\beta_2}{2}\frac{\tilde{h}_1}{\tilde{h}_2}t'^{l}\frac{\partial\tilde{h}_2}{\partial w} = -(l-1)\tilde{u}_2. \quad (43)$$

We do not discuss further reduction of the reduced homogeneous systems associated with $kV_1 + V_2 + V_3$ and $V_1 + V_3$ as they are similar to the ones in [1].

6. Simulation results and discussions

Some conditions can be imposed in the above systems to further reduce them and seek some (very) special and restricted analytical solutions. For example, we can choose the parameter values for k, l, β_1, and β_2 so that some terms drop out. Nevertheless, we continue with the system as it is and solve it numerically to access the performance of the reduced system of ODEs. This will provide an overall picture of the physical process, mainly, the coupled dynamics of the solid and fluid phase velocities and the flow heights. Since the different optimal systems are associated with fundamentally and quantitatively different physical aspects, their qualitative behaviors can be similar as they represent the solution of the same physical system of equations (6)–(7) governing the down-slope mass flows as a mixture of sediment particles and viscous fluid. So, it suffices to discuss the numerical simulations of a representative optimal system. Without loss of generality, we choose the system (34). The results are presented for different solid and fluid pressure parameters β_1 and β_2. Further results are presented for the optimal Lie parameters k and (b_1, b_2). The physical parameters (β_1, β_2) characterize the dynamical system (6)–(7) and the parameters (k, b_1, b_2) are associated to the Lie symmetry and its optimal structure. Since $l = b_2/b_1$ always appears in combination, there are essentially two physical pressure parameters (β_1, β_2) and two Lie parameters (k, l) that govern the system (34). The flow is released from the top-left with the initial flow heights and velocities as indicated in the simulation Figs. 1–3. This corresponds to the mass release from a silo as in Pudasaini et al. [20] and in Dominik and Pudasaini [32]. In what follows (u_s, u_f, h_s, h_f) represent the similarity forms $(\tilde{u}_s, \tilde{u}_f, \tilde{h}_s, \tilde{h}_f)$ and x represents the similarity variable v in (34).

Fig. 1 shows the dynamics and interactions of the solid- and fluid-phase heights (left) and velocities (right) as given by the solution of the system (34). Increasing fluid pressure parameter β_2 results in decrease of the respective flow heights and increase of the flow velocities. In general, these results are consistent with the physics of coupled two-phase mass flow, because for the downslope shear flow of granular material, as flow thins, the velocity generally increases. The phase flow depths and flow velocities evolve non-linearly. After the release of the debris mass, the flow depth first decreases rapidly then slowly in the farther downstream. Similarly, the velocities first increase quickly just below the silo gate and then slowly in the farther downstream. As the fluid pressure parameter increases from top to the bottom panels, the difference between the solid and fluid phase heights increases, so does the difference between the solid and fluid phase velocities. In general, fluid heights are smaller than the solid heights, and the associated fluid velocities are higher than the solid velocities. These are observable phenomena in two-phase debris motion, because higher fluid pressure parameter makes fluid weaker than solid. As the homogeneous PDEs (6)–(7) are explicitly driven by the hydraulic pressure gradient, the higher pressure gradients (either for solid, or fluid, or both) result in larger driving forces leading to the higher flow velocities for the corresponding phases, and thus, the decreased flow depths as the mixture mass moves down slope. These physical aspects are clearly captured by the solutions presented in Fig. 1.

Next, we discuss the effect of the optimal Lie parameters on the flow dynamics. Fig. 2 indicates as k increases the solid flow heights increase. However, the fluid heights remain almost unchanged. Consequently, due to significantly reduced shearing, the solid could not thin further resulting in an increased flow height. Interestingly, within the range of the chosen flow domain and parameters, the dynamics are much less sensitive to the fluid-phase than for the solid-phase. Small change in k results in large change in solid dynamics (flow heights and velocities).

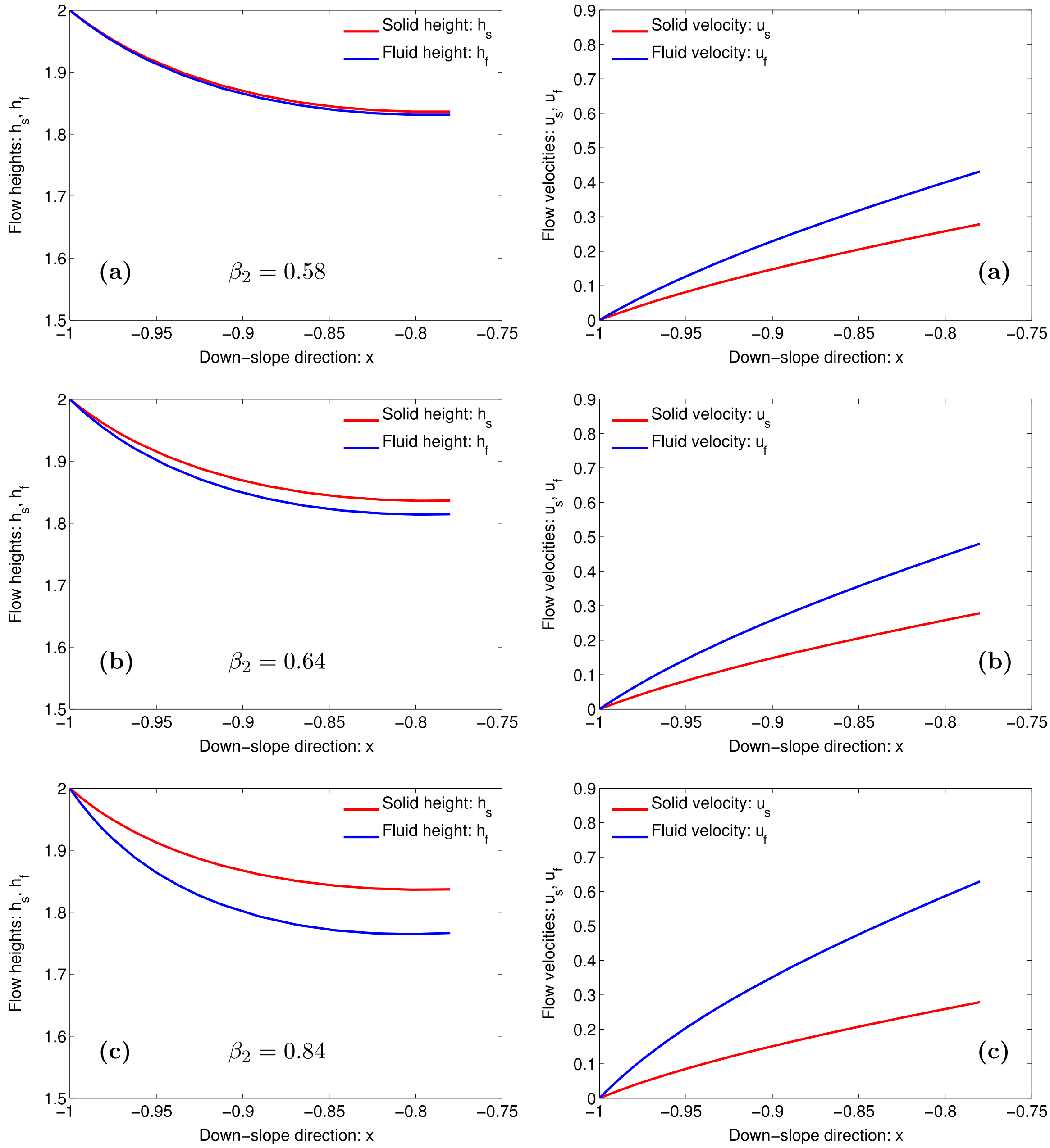

Fig. 1. Effect of the fluid pressure parameters β_2 on the flow dynamics when $\beta_1 = 0.25$ and the Lie parameters $b_1 = 3$, $b_2 = -1.2$, and k=1: the panels show the dynamics and interactions of the solid- and fluid-phase heights (left) and velocities (right).

For larger values of k, the solid dynamics tends to cease while the fluid dynamics continues without much change. This is the situation for a debris release in a gentle slope for which the solid mass moves and deforms very slowly due to (basal) friction. However, as the fluid is weak it deforms and moves quickly along the slope. These are realistic results. For small k, both the solid and fluid after traveling a certain distance along the slope the flow heights do not change so much and tend to maintain constant heights. But due to the inclined channel the flow accelerates and velocities increase continuously along the channel slope. These are observable phenomena in geophysical mass flows [20,17]. Nevertheless, for larger k (=2), as solid moves very slowly the flow depth remains virtually unchanged. This behavior is similar to the creeping as often observed in rock glaciers and frictional granular flows in gentle slope. So, k is somehow associated with the frictional strength of the solid phase. As in Fig. 1, fluid velocities are larger than the solid velocities resulting in larger heights for the solid and smaller heights for fluid. Depending on the value of k, both the flow heights and velocities evolve strongly or weakly non-linearly. Further increase of k results in constant values for the solid heights and velocities. Interestingly, independent of the value of k, by setting the solid pressure parameter to zero ($\beta_1 = 0$) the solid height and velocity remain constant (the initial values), not shown here. The same holds true for the fluid, i.e., by setting the fluid pressure parameter to zero ($\beta_2 = 0$) we obtain the constant initial solutions for fluid height and velocity. This conforms the consistency of the obtained similarity solution, because as the flows are driven by the hydraulic pressure parameters, the motion must cease as these parameters become zero.

Further, we present the results associated with the effect of the optimal Lie parameter b_2 on the flow dynamics. Fig. 3 shows the decreasing value of the parameter b_2 results in the complete change in the flow dynamics (both the flow heights and velocities) more for the fluid and less for the solid. Here, as the value of b_2 decreases the flow heights decrease first rapidly in the vicinity of the flow release (silo gate), then slowly in the farther downslope leading to the more

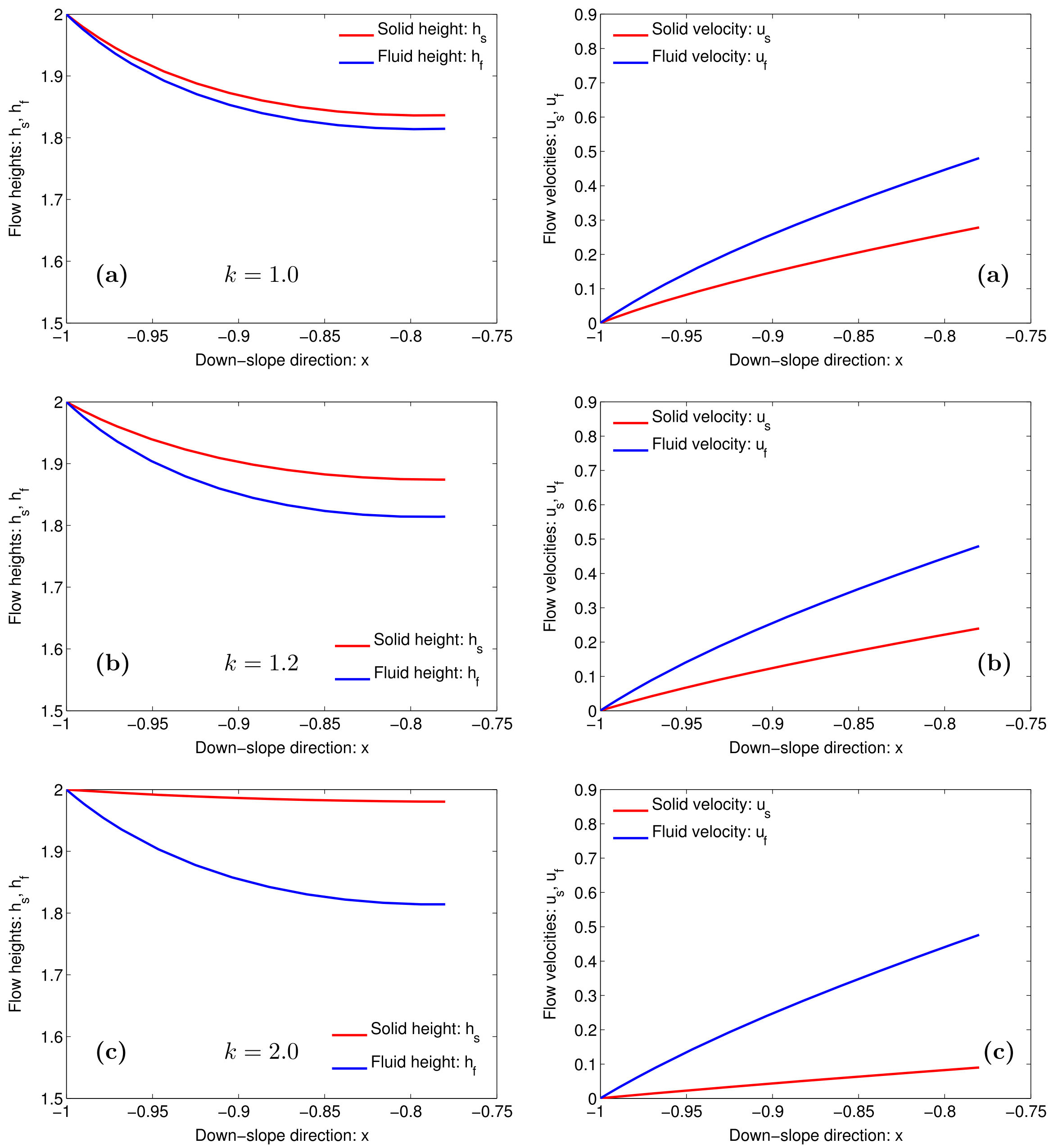

Fig. 2. Effect of the optimal Lie parameter k, the Lie parameters $(b_1, b_2) = (3, -1.2)$, and the solid and fluid pressure parameters $(\beta_1, \beta_2) = (0.25, 0.64)$ on the flow dynamics: The panels show the dynamics and interactions of the solid- and fluid-phase heights (left) and velocities (right).

constant flow heights. Similarly, the flow velocities first increase quickly close to the silo gate, then relatively slowly in the far downstream. The differences between the solid and fluid heights, and the solid and fluid velocities increase quickly after the flow release. This resulted in the rapid drop of the flow heights in the upslope that decreased gently in the farther downslope along the channel. Another interesting aspect is that as the value of b_2 decreases the fluid flow dynamics turns quickly from highly non-linear (in the vicinity of flow release) to weakly non-linear (farther downslope) for the flow depths, whereas the fluid velocities show increasingly non-linear state. So, the flow dynamics are fundamentally different in the above three figures. Simulations reveal that the fluid dynamics is more sensitive to the parameter b_2, whereas the solid dynamics is more sensitive to the parameter k. However, both the solid and fluid phase dynamics are equally influenced by the solid and fluid pressure parameters (β_1, β_2). In all the simulations, the fluid velocities dominate the respective solid velocities as soon as the flow is released. The difference may reduce or increase depending on the parameter selection. For all the flow situations, solid flow heights are dominating the fluid flow heights which is expected. Because, for the sheared inclined channel flow, the higher flow velocities correspond to the lower flow heights, it is consistent with the physics of coupled two-phase mass flows. Larger fluid velocities are expected in the two-phase mass flows down the slope because the fluid is mechanically relatively weak as compared to the solid-phase in the mixture. In Figs. 1 and 2, the phase velocities (mostly) tend to saturate in the farther down-slope, whereas in Fig. 3 the solid and fluid phases accelerate as these phases rapidly thin in the far downstream. Both of these situations are possible in the mass flows. So, from the physical point of view the results presented in Figs. 1–3 are meaningful.

Finally, we would like to point out that usually the solutions obtained by group analysis could be inferred by looking at the equations if the model equations have been sufficiently analyzed and, most, if not all, solutions have already been constructed. This can be

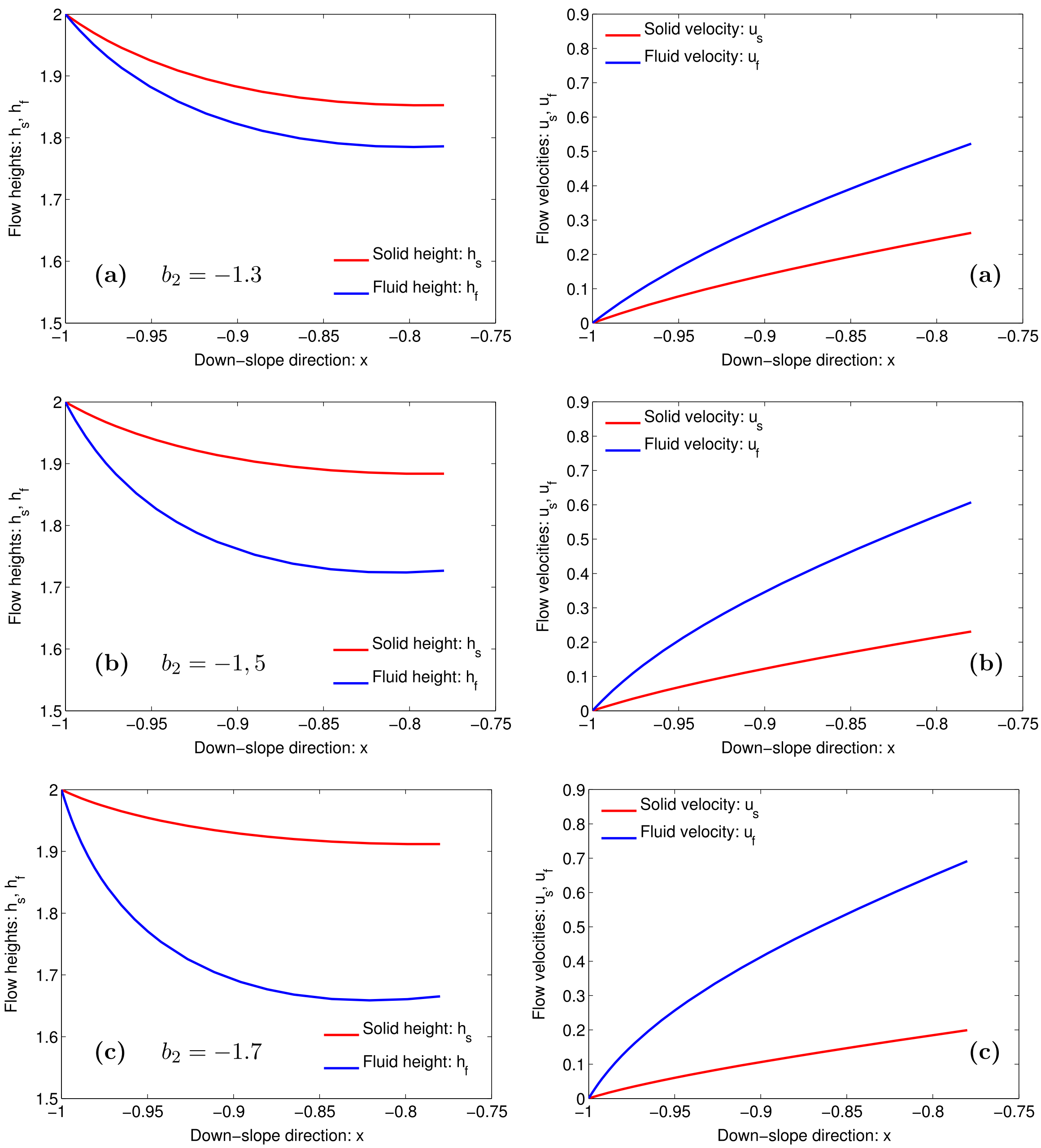

Fig. 3. Effect of the Lie parameter b_2, the Lie parameters $(b_1, k) = (3, 1)$, and the pressure parameters $(\beta_1, \beta_2) = (0.25, 0.64)$ on the flow dynamics: the panels show the dynamics and interactions of the solid- and fluid-phase heights (left) and velocities (right).

the case for the single-phase fluid dynamical equations for which several analytical and exact solutions have already been developed. This includes the pressure driven Euler, viscous Navier–Stokes, and shallow water equations for a fluid. However, our system of PDEs deals with model equations for the flow of a strongly coupled mixture material consisting of the non-Newtonian viscous fluid and frictional granular material. Furthermore, in our model, there are strong interactions between the solid and fluid phases, e.g., due to buoyancy, hydraulic pressure gradients, and friction.

7. Summary

In this paper, we applied the Lie symmetry method to the two-phase mass flow model [18] and constructed some optimal systems of subalgebras corresponding to this system of non-linear PDEs. An optimal system provides precise insights into all possible invariant solutions, hence it is of great importance from the mathematical point of view as well as constraining the system for the physical and engineering applications. For different solutions emerging from infinitesimal symmetries, it is enough to use an optimal system of Lie subalgebras as it contains information about different types of invariant solutions. In particular, this allows to partition all the possible invariant solutions into disjoint sets.

We constructed one-, two-, and three-dimensional optimal systems of Lie subalgebras of the physical system. To construct optimal systems of Lie subalgebras, we first computed one-dimensional ones and follow an inductive procedure. More precisely, we constructed the other optimal systems by computing all one dimensional higher Lie subalgebras and removing the redundancies by the action of a general adjoint operator. This involved an analysis of equivalence classes of these Lie subalgebras, and resulted in an optimal system by considering only one representative from each equivalence class. We reduced the two-phase mass flow system of PDEs into another systems of PDEs. Using the fact that the Lie bracket contains information about further reduction, we further reduce to systems of ODEs and PDEs.

We solved a system numerically and analyzed in detail with its implications, in particular presented simulation results for flows that correspond to the mass release from a silo. This provided an overall picture of the physical process, the coupled dynamics of the solid and fluid phase velocities and the flow heights. The results are presented for different solid and fluid pressure parameters and Lie parameters. The physical parameters characterize the dynamical system, whereas the other parameters are associated to the optimal structure of the Lie symmetry. Simulations reveal that the fluid dynamics is more sensitive to the Lie parameter b_2, whereas the solid dynamics is more sensitive to another Lie parameter k. However, both solid and fluid phase dynamics are equally influenced by the solid and fluid pressure parameters (β_1, β_2). Increasing fluid pressure parameter results in decrease of the respective flow heights and increase of the flow velocities. Higher pressure gradients result in higher flow velocities, and hence, it decreases flow depths as the mixture mass moves down slope. Zero pressure parameters result in the constant initial solutions for both solid and fluid heights and velocities. These physical aspects are clearly captured by the simulations of the optimal system. This confirms the consistency of the obtained similarity solutions.

In all the simulations, the fluid velocities dominate the respective solid velocities as soon as the flow is released. The difference may reduce or increase depending on the parameter selection. For all flow situations in consideration, solid flow heights are dominating fluid flow heights which is expected. Because, for the sheared inclined channel flow, higher flow velocities correspond to lower flow heights, consistent with the physics of coupled two-phase mass flows. Larger fluid velocities are expected in the two-phase mass flows down the slope, because the fluid is mechanically relatively weak as compared to the solid-phase in the mixture. The phase heights (mostly) tend to saturate in the farther down-slope, whereas the phases mostly accelerate as the flow rapidly thins in the far downstream. These situations are possible in mass flows. So, from the physical point of view the results presented here are meaningful.

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