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# On analytical solutions of a two-phase mass flow model[☆]

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ABSTRACT

We investigate a two-phase mass flow model by constructing analytical solutions with their physical significance. We use the method of splitting and separation of variables to reduce the system of non-linear PDEs modeling the two-phase mass flow into quasilinear PDEs. In particular, the system of non-linear PDEs is reduced into Riccati equations and Burgers equations, thereby making it possible to solve. Starting with simple analytical solutions, we construct analytical solutions with increased complexities for the phase velocities and the phase heights as functions of space and time. Furthermore, we use the Lie group action to generate more analytical solutions and analyze their possible invariance. We also present a perspective called relative non-invariance associated to the underlying physics relevant to multi-phase flows, namely, the relative velocity and relative flow depths between the phases. Finally, we present detailed analysis and discussion on the time and spatial evolutions of the analytical solutions for solid and fluid phase velocities and the flow depths. The obtained analytical solutions corroborate with the physics of two-phase mass flows down a slope.

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## 1. Introduction

Debris flows are two-phase, gravity-driven mass flows consisting of solid particles and fluid. Debris flows play an important role in environment, geophysics and engineering [1–11]. Two-phase flows are generally characterized by the relative motion between the solid and fluid phases, which depends on the mixture composition, solid–fluid interactions, and the dynamics as modeled by the main driving forces. These flows are extremely destructive and dangerous in nature, hence, predicting the nature of the flow, its dynamics, and runout distances could be very valuable. The solid and fluid phase velocities in debris flows may deviate substantially from each other essentially affecting the flow mechanics. This makes debris flows a challenging research area. There has been extensive field, experimental, theoretical, and numerical investigations on the dynamics, consequences, and industrial applications of such mixture flows [11–17].

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Various research in the past few decades focused on different aspects of two-phase debris flows [1–9]. More recently, in Pudasaini (2012) [18], two-phase mass flows have been studied with a comprehensive model that explicitly includes different interactions between the solid particles and the viscous fluid. The model in [18] constitutes the most generalized two-phase flow model to date. This model reproduces previous simple models, which was considered for single- and two-phase avalanches and debris flows, as special cases. Recently, extensive simulations and applications of the general two-phase mass flows have been carried out for different mass flows including landslide impacting a reservoir [19], glacial lake outburst floods [20], and complex and multiple process chains with breaching of dam resulting in subsequent debris floods [17]. Similarly, several analytical solutions have been constructed for two-phase or reduced mass flows [21–24]. Exact and analytical solutions of the complex two-phase mass flow models are very important as these solutions highlight the insight of the underlying physical model. Furthermore, these exact solutions can be utilized to calibrate and validate the numerical schemes and numerical simulations before such simulations methods can be applied to real world problems. For these reasons, here, we are concerned in constructing new analytical solutions for the simplified form of the two-phase mass flow model [18].

In this paper, we advance the investigation of the two-phase mass flow model from [18,23] by constructing several families of new explicit solutions and studying their physical significance. We apply the method of splitting and method of separation of variables to solve the system into consideration. The method of splitting allows us to reduce the two-phase mass flow model into a decoupled system of first order quasilinear PDEs. Further, we reduce these quasilinear PDEs into Burgers equations and Riccati equations. This allows us to construct solutions of the two-phase mass flow model. Analytical solutions are obtained with increased complexities for the phase velocities and the phase heights in two-phase mixture mass flows as functions of space and time. Furthermore, these solutions for flow velocities and depth evolutions are shown to be in line with the physics of two-phase mass flows down a channel.

The Lie symmetry method is a widely used tool to study differential equations arising from wide range of physical problems. Its application includes the reduction of differential equations, order reduction of ordinary differential equations, construction of invariant solutions, and mapping between the solutions in mechanics, applied mathematics and mathematical physics, and applied and theoretical physics [10,25–32]. In [23], Ghosh Hajra et al. initiated studying the model in [18] using the Lie symmetry method where the most general symmetry Lie algebra of the system is computed. Moreover, several physically significant analytic and numerical solutions are presented. More recently, in [24], the optimal systems of the symmetry Lie algebra has been used to reduce the model (from [23]) into other systems of PDEs and ODEs. Also, these optimal Lie subalgebras are used to generate several physically relevant numerical solutions. In this paper, we use the Lie symmetry algebra constructed in [23] to generate Lie symmetry transformed solutions [25,31] from the newly obtained solutions. From the viewpoint of the number of parameter describing solutions, some transformed solutions are invariant while others are not. Furthermore, based on the structure of transformed solutions and the difference between the pair of solid and fluid phase velocities, we introduce a notion called relative non-invariance mapping.

## 2. Two-phase mass flow model

In this section, we briefly review the basic features of a two-phase mass flow model into consideration, as in Ghosh Hajra et al. [23,24]. This two-phase mass flow model is the one-dimensional inclined channel flow model of [11], which is a special case of the general model in [18]. Let $t$ be time, $X$ and $Z$ be coordinates along and normal to the slope with inclination $\zeta$. In the following, the solid and fluid constituents are denoted by the suffices $s$ and $f$, respectively. Let $h$ be the mixture flow depth, $h_s = \alpha_s h, h_f = \alpha_f h$ be the solid and fluid flow depths, and $Q_s = h_s u_s = \alpha_s h u_s, Q_f = h_f u_f = \alpha_f h u_f$ are the corresponding fluxes, where $\alpha_s$, $\alpha_f (= 1 - \alpha_s)$ are the solid and fluid volume fractions, respectively. The solid and fluid net driving forces are denoted by $S_s$ and $S_f$ and are given by $S_s = \sin\zeta - \tan\delta(1-\gamma)\cos\zeta$, and $S_f = \sin\zeta$, where $\delta$ is the friction angle and $\gamma = \rho_f/\rho_s$ is the density ratio between the fluid and solid material densities.

The depth-averaged mass and momentum conservation equations for the solid and fluid phases [23,24] are

$$\frac{\partial h_s}{\partial t} + \frac{\partial Q_s}{\partial X} = 0, \quad \frac{\partial h_f}{\partial t} + \frac{\partial Q_f}{\partial X} = 0, \tag{1}$$

$$\begin{aligned} &\frac{\partial Q_s}{\partial t} + \frac{\partial}{\partial X}\left(Q_s^2 h_s^{-1}\right) + \frac{\partial}{\partial X}\left(\frac{\beta_s}{2} h_s\left(h_s + h_f\right)\right) = h_s S_s, \\ &\frac{\partial Q_f}{\partial t} + \frac{\partial}{\partial X}\left(Q_f^2 h_f^{-1}\right) + \frac{\partial}{\partial X}\left(\frac{\beta_f}{2} h_f\left(h_f + h_s\right)\right) = h_f S_f, \end{aligned} \tag{2}$$

where $\beta_s = \varepsilon K p_{b_s}$, $\beta_f = \varepsilon p_{b_f}$, $p_{b_f} = \cos\zeta$, $p_{b_s} = (1-\gamma)p_{b_f}$, $L$ and $H$ denote the typical length and depth of the flow with the aspect ratio $\varepsilon = H/L$. The earth pressure coefficient $K$ and $\tan\delta$ include frictional behavior of the solid-phase. Here, $p_{b_f}$ and $p_{b_s}$ are associated with the effective basal fluid and solid pressures, $\beta_s, \beta_f$ are the hydraulic pressure parameters associated with the solid- and the fluid-phases respectively, and $(1-\gamma)$ indicates the buoyancy reduced solid normal load.

For notational simplification, we introduce the transformation as in Ghosh Hajra et al. (2015) [23]:

$$x = X - S_s t^2/2,\ y = X - S_f t^2/2, \quad \hat{Q}_s = Q_s - h_s S_s t,\ \hat{Q}_f = Q_f - h_f S_f t. \tag{3}$$

The variables $x$ and $y$ are the moving spatial coordinates for the solid and fluid respectively. They are considered as independent variables as $S_s$ and $S_f$ are independent. Using (3), Eqs. (1)–(2) can be transformed into a homogeneous system of partial differential equation [23,24]:

$$\begin{aligned} &\frac{\partial h_s}{\partial t} + \frac{\partial \hat{Q}_s}{\partial x} = 0, \\ &\frac{\partial h_f}{\partial t} + \frac{\partial \hat{Q}_f}{\partial y} = 0, \end{aligned} \tag{4}$$

$$\begin{aligned} &\frac{\partial \hat{Q}_s}{\partial t} + \frac{\partial}{\partial x}\left(\hat{Q}_s^2 h_s^{-1}\right) + \frac{\partial}{\partial x}\left(\frac{\beta_s}{2} h_s\left(h_s + h_f\right)\right) = 0, \\ &\frac{\partial \hat{Q}_f}{\partial t} + \frac{\partial}{\partial y}\left(\hat{Q}_f^2 h_f^{-1}\right) + \frac{\partial}{\partial y}\left(\frac{\beta_f}{2} h_f\left(h_f + h_s\right)\right) = 0. \end{aligned} \tag{5}$$

Next, we replace $\hat{Q}_s$ by $\hat{u}_s h_s$ and $\hat{Q}_f$ by $\hat{u}_f h_f$, and introduce the suffices $1 := s, 2 := f$ for the variables and parameters associated with the solid and fluid components respectively. After dropping the hats from the resulting system, we obtain

$$\begin{aligned} &\frac{\partial h_1}{\partial t} + \frac{\partial (u_1 h_1)}{\partial x} = 0, \\ &\frac{\partial h_2}{\partial t} + \frac{\partial (u_2 h_2)}{\partial y} = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} &\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + \beta_1\frac{\partial h_1}{\partial x} + \frac{\beta_1}{2}\frac{\partial h_2}{\partial x} + \frac{\beta_1}{2}\frac{h_2}{h_1}\frac{\partial h_1}{\partial x} = 0, \\ &\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial y} + \beta_2\frac{\partial h_2}{\partial y} + \frac{\beta_2}{2}\frac{\partial h_1}{\partial y} + \frac{\beta_2}{2}\frac{h_1}{h_2}\frac{\partial h_2}{\partial y} = 0. \end{aligned} \tag{7}$$

The fourth and fifth terms associated with $\beta/2$ in (7) that emerge from the pressure-gradients, include buoyancy (through $1-\gamma$), friction (through $K$, and $\tan\delta$), and net driving forces ($S_s, S_f$). These forces have distinct mechanical significance in explaining the physics of the two-phase mass flows that are not present in previous models [23,24].

In [23], the Lie symmetry method has been used to analyze the model (6)–(7), the most general symmetry Lie algebra of the system is computed and a simple Lie symmetry transformation is used to construct many physically significant analytic and numerical solutions. More recently, in [24], the optimal systems of the symmetry Lie algebra has been constructed. These optimal systems have been used to study the model (6)–(7), in particular, many systems of reduced PDEs and ODEs have been constructed. Furthermore, many physically relevant numerical solutions are obtained. Here, we construct some new analytical solutions for the phase velocities and phase heights for the system (6)–(7). Moreover, by applying the action of the Lie group we transform the obtained solutions and analyze their invariance. In doing so, we present a novel perspective and definition of a relative non-invariance transformation associated to the underlying physics relevant to multi-phase flows.

## 3. Reduction to Burgers equation and some simple solutions

Here we will use the method of separation of variables to produce some solutions of the system (6)–(7). To begin with, in this section, we construct a family of simple solutions by making the simplifying assumption that $h_i, i = 1, 2$ do not depend on the spatial variables $x$ and $y$. Later, we will relax this assumption and generate more complex solutions. The main motivation behind this assumption is that it allows to decouple (6)–(7) into two independent systems for solid and fluid:

$$\frac{\partial h_1}{\partial t} + h_1 \frac{\partial u_1}{\partial x} = 0, \ \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = 0, \tag{8}$$

$$\frac{\partial h_2}{\partial t} + h_2 \frac{\partial u_2}{\partial x} = 0, \ \frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial y} = 0. \tag{9}$$

Note that second equations of (8) and (9) are inviscid Burgers equation [33] which can be solved using various methods such as separation of variables and method of characteristics [34]. To illustrate the method of separation of variables, we solve the second equation of (8). Assume that $u_1(x, t) = f(x)p(t)$. Then,

$$\frac{\partial u_1}{\partial t} = f(x)p'(t), \ \frac{\partial u_1}{\partial x} = f'(x)p(t). \tag{10}$$

Substituting (10) in the second equation of (8), we get

$$f(x)\left(f'(x)p(t)^2 + p'(t)\right) = 0.$$

To use the method of separation of variables, we assume that $f'(x)$ is a constant. More precisely, if we can find $f(x)$ and $p(t)$ such that

$$f'(x) = a, \tag{11}$$

$$ap(t)^2 + p'(t) = 0, \tag{12}$$

then, we have a solution of the form $u_1(x, t) = f(x)p(t)$ of the second equation of (8), where $f(x) = ax + c$ and $p(t) = \frac{1}{at+b}$. In other words, $u_1(x, t) = \frac{ax+c}{at+b}$ is a solution of (8). Now, substituting $u_1$ in the first equation of (8) and using the assumption that $h_1$ does not depend on the spatial variables, we get

$$\frac{\partial h_1}{\partial t} + \frac{ah_1}{at + b} = 0. \tag{13}$$

This equation has a solution of the form $h_1 = \frac{d}{at+b}$. In these derivations $b, c, d$ are constants of integrations.

To summarize, we showed

$$h_1 = \frac{d_1}{a_1 t + b_1} \quad \text{and} \quad u_1 = \frac{a_1 x + c_1}{a_1 t + b_1},$$

solves the system (8) and

$$h_2 = \frac{d_2}{a_2 t + b_2} \quad \text{and} \quad u_2 = \frac{a_2 x + c_2}{a_2 t + b_2},$$

solves the system (9). In particular, we proved the following lemma.

**Lemma 3.1.** *The quadruple* $\left(h_1 = \frac{d_1}{a_1 t+b_1}, u_1 = \frac{a_1 x+c_1}{a_1 t+b_1};\ h_2 = \frac{d_2}{a_2 t+b_2}, u_2 = \frac{a_2 x+c_2}{a_2 t+b_2}\right)$ *of functions solves the system (6)–(7) where* $a_i, b_i, c_i, d_i; i = 1, 2$ *are some constants.*

The physical values of the constants $a_i, b_i, c_i$ and $d_i$ can be constrained by imposing either initial or boundary conditions to the system (6)–(7).

## 4. Reduction to Burgers and Riccati type equations via splitting and their solutions

In this section, we construct solutions by relaxing the simplifying assumption made in Section 3. More precisely, we assume only one of $h_1$ or $h_2$ does not depend on the spatial variables and for the concreteness, assume $h_2$ does not depend on $y$. Our approach here is to split the problem of solving the system into the problem of solving various first order quasilinear (linear) equations. We illustrate this by solving the second equation of (7) which, due to the simplifying assumptions, becomes

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial y} + \frac{\beta_2}{2} \frac{\partial h_1}{\partial y} = 0. \tag{14}$$

If we succeed to find $u_2$ and $h_1$, and a new auxiliary function $w(y)$ such that

$$(a_1 t + b_1) \frac{\partial u_2}{\partial t} + (a_1 t + b_1) u_2 \frac{\partial u_2}{\partial y} + d_1 w(y) = 0, \tag{15}$$

and

$$\frac{\beta_2}{2} \frac{\partial h_1}{\partial y} = \frac{d_1 w(y)}{a_1 t + b_1}, \tag{16}$$

where $a_1, b_1$ and $d_1$ are constants, then we have succeeded in finding $(u_2, h_1)$ satisfying (14). In other words, we solve (14) by solving the system (15)–(16) consisting of a pair of quasilinear equations. This is an example of splitting. Next, we discuss how to solve this system.

Let us start with (15) which is a quasilinear PDE and we can solve this by method of separation of variables. We spell this out in detail as it will be used again in the next section. The term without derivative $d_1 w(y)$ suggests the ansatz $u_2 = w(y) g(t)$ for the solution. Using this in (15), we get

$$w(y) \left[ (a_1 t + b_1) \{ g'(t) + w'(y) g(t)^2 \} + d_1 \right] = 0. \tag{17}$$

From (17), we observe if $w'(y) = \alpha$, where $\alpha$ is a constant, then it is possible to implement separation of variables technique. In particular, for such $w(y)$, if $g(t)$ is a solution of the Riccati equation

$$(a_1 t + b_1) g'(t) + \alpha (a_1 t + b_1) g(t)^2 + d_1 = 0, \tag{18}$$

then $u_2 = w(y) g(t)$ is a solution of (15). Unlike usual Riccati equations the coefficient in the 0th order term is $d_1/(a_1 t + b_1)$. This makes exact solutions much more interesting (see, Section 7). Riccati equations play important role in physics and mathematics in constructing exact solutions [35,36].

The following lemma summarizes these discussions and derivations.

**Lemma 4.1.** *Let $u_2(y,t) = (\alpha y + \delta)g(t)$, then $u_2$ is a solution of the quasilinear PDE*

$$(a_1 t + b_1)\frac{\partial u_2}{\partial t} + (a_1 t + b_1)u_2\frac{\partial u_2}{\partial y} + d_1(\alpha y + \delta) = 0.$$

**Proof.** $(a_1t+b_1)\frac{\partial u_2}{\partial t}+(a_1t+b_1)u_2\frac{\partial u_2}{\partial y}+d_1(\alpha y+\delta) = (\alpha y+\delta)((a_1t+b_1)g'(t)+\alpha(a_1t+b_1)g(t)^2+d_1) = 0.$ □

Next, we find $h_1$. Let $u_2$ be as in Lemma 4.1. From (16),

$$\frac{\beta_2}{2}\frac{\partial h_1}{\partial y} = \frac{d_1(\alpha y + \delta)}{a_1 t + b_1}. \tag{19}$$

Integrating with respect to $y$ yields

$$h_1 = \frac{d_1(\alpha y^2/2 + \delta y + f)}{\beta_2(a_1 t + b_1)}. \tag{20}$$

**Lemma 4.2.** *Let $u_2$ be as in Lemma 4.1. Then, $h_2 = Ke^{-\alpha\int g(t)\,dt}$ satisfies the second equation of (6), where $K$ is a constant.*

**Proof.** Using $u_2$ in the second equation of (6), we get

$$\frac{\partial h_2}{\partial t} + \alpha g(t) = 0.$$

Obviously, $h_2 = Ke^{-\alpha\int g(t)\,dt}$ is a solution of this equation. □

Now, we are ready for new sets of the extended solutions of the system (6)–(7).

**Proposition 4.3.** *Let $u_2 = (\alpha y + \delta)g(t)$ as in Lemma 4.1, $h_1 = \frac{d_1(\alpha y^2/2+\delta y+f)}{\beta_2(a_1t+b_1)}$, $h_2 = Ke^{-\alpha\int g(t)\,dt}$ as in Lemma 4.2, and $u_1 = \frac{a_1x+c_1}{a_1t+b_1}$, then $(u_1, h_1, u_2, h_2)$ is a solution of (6)– (7).*

**Proof.** We have already shown $h_1, u_2$ and $h_2$ satisfy the second equations of (6) and (7). We only need to show $u_1$, $h_1$ and $h_2$ satisfy the first equations of (6) and (7). As $h_1$ and $h_2$ are independent of $x$, the first equation of (7) becomes the Burgers equation

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} = 0,$$

and obviously $u_1 = \frac{a_1x+c_1}{a_1t+b_1}$ is a solution of this equation as we have already seen. Now, it remains to check $u_1$ and $h_1$ satisfy the first equation of (6). Since

$$\frac{\partial h_1}{\partial t} = -\frac{a_1 h_1}{a_1 t + b_1}, \quad \text{and} \quad \frac{\partial(u_1 h_1)}{\partial x} = h_1\frac{\partial u_1}{\partial x} = \frac{a_1 h_1}{a_1 t + b_1},$$

the first equation of (6) is satisfied. This completes the proof of the proposition. □

**Remark 1.** In principle, we could take $K = K(x)$ in $h_2$. In this case, we can solve the system (6)–(7) with some constraints on $K(x)$. Consequently, this general form of $h_2$ has the effect that the expression for $u_1$ from Proposition 4.3 will change substantially.

For the sake of clarity in what follows, let us summarize the key steps used in this section. First, we used the separation of variable technique to split the given system into several first order quasilinear (linear)

PDEs and then, solved the first order quasilinear (linear) PDEs using various techniques, such as the method of characteristics (see Section 5). For example, we split the second equation of (7) into quasilinear PDE (15) and a linear PDE (16) and solved these PDEs. The solutions satisfying all the various quasilinear (linear) equations put together gave rise to a solution of the given system. Splitting is a common practice in mathematics and physics which helps to reduce the given problem into relatively easier problems, and understanding of such reduced problems give insights on solutions of the given problem.

## 5. More non-trivial solutions

In this section, we improve Proposition 4.3 and construct complex solutions following similar strategy. Let $u_2, h_1$ be as in Proposition 4.3 and $h_2$ be as in Lemma 4.2 but $K = K(x)$, rather than just a constant. Our goal here is to find a more general $u_1$ so that we have further more general solutions of the system (6)–(7). Now, substituting $h_2$ in the first equation of (7) gives rise to the following equation for $u_1$:

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{\beta_1}{2} K'(x) e^{-\alpha \int g(t)\, dt} = 0. \tag{21}$$

Eq. (21) is a first order non-homogeneous quasilinear PDE and a general solution can be obtained using the method of characteristics for any $K(x)$. We expect that it is possible to solve the system (6)–(7) with a general $K(x)$ but we do not pursue this direction in this paper. It will be interesting to investigate further. Here, we will instead concentrate on a particular type of $K(x)$, namely, linear functions and use them to construct solutions of (6)–(7).

From now on, we fix $K(x) = px + q$ and solve (21). In this case, a solution of (21) together with the choices of $(u_1, u_2, h_2)$ as above provides a solution. For notational simplicity, we define

$$\gamma(t) = \frac{\beta_1}{2} K'(x) e^{-\alpha \int g(t)\, dt} = \frac{p\beta_1}{2} e^{-\alpha \int g(t)\, dt}.$$

The characteristics equation of the quasilinear equation (21) is

$$dt = \frac{dx}{u_1} = \frac{du_1}{-\gamma(t)}. \tag{22}$$

Integrating $dt = \frac{du_1}{-\gamma(t)}$ of (22),

$$C_1 = u_1 + \int \gamma(t)\, dt, \tag{23}$$

where $C_1$ is a constant of integration. Using $u_1$ from (23) to $dt = \frac{dx}{u_1}$ part in (22), we get

$$dt = \frac{dx}{C_1 - \int \gamma(t)\, dt},$$

and integration of this equation gives rise to

$$x = C_1 t - \int \left[ \int \gamma(t)\, dt \right] dt + C_2,$$

where $C_2$ is a constant of integration. With the help of (23), we can rewrite this as

$$C_2 = x - t u_1 - t \int \gamma(t)\, dt + \int \left[ \int \gamma(t)\, dt \right] dt. \tag{24}$$

By the general theory of quasilinear PDE [34], a general solution of (21) is given by a two parameter family (function)

$$F(C_1, C_2) = 0. \tag{25}$$

An arbitrary $F(C_1, C_2)$ may not serve the purpose of solving the system (6)–(7), we must choose $F(C_1, C_2)$ wisely. To make such a choice, we decrease degrees of freedom to choose $F$ by requiring $u_1$ to have the form $u_1 = \frac{a_1 x + c_1}{a_1 t + b_1} + \tau(\gamma(t))$. Note that such $u_1$ generalizes the $u_1$ obtained in Proposition 4.3. To get an idea to pick an appropriate $F$, first let us consider a relatively simple case namely assume $\gamma(t)$ is identically zero. In this case, we get $u_1 = \frac{a_1 x + c_1}{a_1 t + b_1}$ as in Proposition 4.3, if we choose $F(C_1, C_2) = b_1 C_1 - a_1 C_2 - c_1$. This suggests similar $F$ could be used for the general case ($\gamma(t)$ is necessarily not identically zero). It turns out that, in the general case at hand, the same $F(C_1, C_2) = b_1 C_1 - a_1 C_2 - c_1$ leads to

$$u_1 = \frac{a_1 x + c_1}{a_1 t + b_1} - \int \gamma(t)\, dt + \frac{a_1}{a_1 t + b_1} \int \left[ \int \gamma(t)\, dt \right] dt, \tag{26}$$

which is indeed the type of $u_1$ we were looking for.

**Proposition 5.1.** *Let $u_2 = (\alpha y + \delta) g(t)$ as in Lemma 4.1, $h_1 = \frac{d_1(\alpha y^2/2 + \delta y + f)}{\beta_2(a_1 t + b_1)}$, $h_2(x,t) = (px + q) e^{-\alpha \int g(t)\, dt}$ and $u_1 = \frac{a_1 x + c_1}{a_1 t + b_1} - \int \gamma(t)\, dt + \frac{a_1}{a_1 t + b_1} \int \left[\int \gamma(t)\, dt\right] dt$. Then, $(u_1, h_1, u_2, h_2)$ solves the system (6)–(7).*

**Proof.** By construction, $h_1$, $u_2$ and $h_2$ satisfy the second equation of (6) and (7). Moreover,

$$\frac{\partial u_1}{\partial t} = -u_1 \frac{\partial u_1}{\partial x} - \gamma(t) = -u_1 \frac{\partial u_1}{\partial x} - \frac{\beta_1}{2} \frac{\partial h_2}{\partial x}, \tag{27}$$

showing

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{\beta_1}{2} \frac{\partial h_2}{\partial x} = 0.$$

This means, $u_1, h_1, h_2$ satisfy the first equation of (7). As in Proposition 4.3, we can check $u_1$ and $h_1$ satisfy the first equation of (6). Hence, this completes the proof. □

**Remark 2.** Another set of solutions of (6)–(7) is given by $u_1 = (\alpha x + \delta) g(t)$, $h_1 = (py + q) e^{-\alpha \int g(t)\, dt}$, $u_2 = \frac{a_2 y + c_2}{a_2 t + b_2} - \int \gamma(t)\, dt + \frac{a_2}{a_2 t + b_2} \int \left[\int \gamma(t)\, dt\right] dt$, $h_2 = \frac{d_2(\alpha x^2/2 + \delta x + f)}{\beta_2(a_2 t + b_2)}$, where $g(t)$ is a solution of the corresponding Riccati equation (18) and $\gamma(t) = \frac{p\beta_2}{2} e^{-\alpha \int g(t)\, dt}$.

There are important implications of the analytical solutions contained in Proposition 5.1. It shows that change in the flow depth of a phase influences the velocity of the other (complementary) phase in the mixture. This implies a strong and direct coupling between the phases in the mixture flow. Further importance of such coupling is that all the solutions for the phase heights, and phase velocities are now the functions of both of space and time. Time evolution of the solutions are constructed in terms of the solution $g(t)$ of a general Riccati-type equation that emerges from the momentum equations of the considered model. Velocity and flow depth solutions are up to quadratic in the space variable, but appear to be very complex combinations of the Bessel's functions of the first and second kinds and the solution of the general Riccati-type equation (see, Section 7). Particularly, due to shearing as $h_2$ thins farther in the downstream, $u_2$ and consequently due to the drift between the phase velocities, $u_1$ becomes larger. This is seen in the third term on the right hand side of $u_1$.

## 6. Action of one parameter groups on solutions

A novel application of Lie symmetry method is that symmetries can be used to generate further solutions from known solutions of the system into consideration. The key observation is that a symmetry group of a system of PDEs maps a solution into another solution. In this section, we generate more solutions from solutions in the previous sections using symmetries of the system (6)–(7). Symmetries of the system (6)–(7) have been studied in [23] by Ghosh Hajra et al. where the largest possible symmetry Lie algebra is computed and shown that it is a five dimensional Lie algebra with a basis given by $\{V_1,\dots,V_5\}$, where

$$\begin{aligned} &V_1=\frac{\partial}{\partial x},\ V_2=\frac{\partial}{\partial y},\ V_3=\frac{\partial}{\partial t},\\ &V_4=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+2h_1\frac{\partial}{\partial h_1}+2h_2\frac{\partial}{\partial h_2}+u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2},\\ &V_5=t\frac{\partial}{\partial t}-2h_1\frac{\partial}{\partial h_1}-2h_2\frac{\partial}{\partial h_2}-u_1\frac{\partial}{\partial u_1}-u_2\frac{\partial}{\partial u_2}. \end{aligned} \tag{28}$$

Let $\mathcal{G}_i=\exp(\varepsilon V_i)$ be the one parameter group [31] associated to $V_i$. Assume $(u_1,u_2,h_1,h_2)$ be a solution of the system of PDEs constructed in the previous sections, then, the groups $\mathcal{G}_i$ transforms this solution to another solution $(\tilde{u}_1,\tilde{u}_2,\tilde{h}_1,\tilde{h}_2)$ as follows.

For $\mathcal{G}_1,\mathcal{G}_2$ and $\mathcal{G}_3$, the actions are translation on $x$-, $y$-, and $t$-directions, respectively. For example, for $\mathcal{G}_1$,

$$\tilde{u}_i=u_i(x-\varepsilon,y,t),\ \ \tilde{h}_i=h_i(x-\varepsilon,y,t).$$

The action of $\mathcal{G}_4$ is by scaling (dilatation) on the spatial variables $(x,y)$ and the dynamical variables $(u_i,h_i)$:

$$\begin{aligned} \tilde{u}_i&=e^{\varepsilon}u_i(e^{-\varepsilon}x,e^{-\varepsilon}y,t),\\ \tilde{h}_i&=e^{2\varepsilon}h_i(e^{-\varepsilon}x,e^{-\varepsilon}y,t). \end{aligned} \tag{29}$$

The action of $\mathcal{G}_5$ is also by dilatation:

$$\begin{aligned} \tilde{u}_i&=e^{-\varepsilon}u_i(x,y,e^{-\varepsilon}t),\\ \tilde{h}_i&=e^{-2\varepsilon}h_i(x,y,e^{-\varepsilon}t). \end{aligned} \tag{30}$$

### 6.1. Stabilizing transformations and new solutions

Our next goal is to analyze the action of these one parameter groups $\mathcal{G}_1,\dots,\mathcal{G}_5$ on the solutions obtained in the previous sections. For this, we first recall the notion of invariant transformation related to a group action. Let $\mathcal{G}$ be a symmetry group of the system (6)–(7) and let us use $S$ to denote the set of all solutions of the system. As $\mathcal{G}$ is a group of symmetries, each $G\in\mathcal{G}$ defines a map $G:S\to S$. Let $X\subset S$, the map induced by a symmetry transformation $G$ stabilizes $X$ if $G(X)\subset X$. Such a map is known as a stabilizer of $X$ [37]. To emphasize the role of $X$, we call such a $G\in\mathcal{G}$ a stabilizing transformation of $X$. Let $G\in\mathcal{G}$ be such that $G$ is not a stabilizing transformation of $X$. Then, there will a solution of the system (6)–(7) in $G(X)$ which is not in $X$. In other words, starting from solutions in $X$ we have found new solutions of the system. This means that if there are symmetry transformations which are not stabilizing transformations of $X$, then they give rise to new solutions of the system (6)–(7).

Let $X_3,X_4$ and $X_5$ denote the set of solutions constructed in Section 3, Section 4 and Section 5 respectively. By construction of the solutions, the solution sets $X_3,X_4$ and $X_5$ are determined by a family of parameters. For example, a general solution which is in $X_3$ has the form $\left(h_1=\frac{d_1}{a_1t+b_1},\ u_1=\frac{a_1x+c_1}{a_1t+b_1},\ h_2=\frac{d_2}{a_2t+b_2},\ u_2=\frac{a_2x+c_2}{a_2t+b_2}\right)$ and such a solution is determined by six free parameters after rearranging the constants appearing in the solution. In other words, $X_3$ is determined by six free parameters. With parameter dependence of the solution sets $X_i$ in mind, we will sometimes refer a stabilizing transformation of $X_i$ as a parameter stabilizing transformation.

Now, let us analyze the action of one parameter groups $\mathcal{G}_i$ on $X_3$. As actions of $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are by translations, these actions transform an element of $X_3$ into another element of $X_3$. This means an individual solution in $X_3$ changes into another solution in $X_3$ while the nature of solutions do not change, which is to say $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ consists of stabilizing transformations of $X_3$. Physically, this means solutions in $X_3$ remain qualitatively and separately invariant under the action of these groups. The same is true for the action of $\mathcal{G}_4$ and $\mathcal{G}_5$.

Next, we investigate the actions on the solutions in $X_4$. It is easy to observe that $\mathcal{G}_1$ and $\mathcal{G}_3$ consists of stabilizing transformations of $X_4$. Furthermore, $u_1$ and $h_2$ are invariant under the action of $\mathcal{G}_2$ as they are independent of $y$. However, $u_2$ and $h_1$ transform interestingly under the action of $\mathcal{G}_2$ as follows:

$$\tilde{u}_2 = (\alpha y + \delta - \alpha\varepsilon)g(t), \quad \tilde{h}_1 = \frac{d_1\left(\frac{\alpha y^2}{2} + (\delta - \alpha\varepsilon)y + (\frac{\alpha\varepsilon^2}{2} - \delta\varepsilon + f)\right)}{\beta_2(a_1 t + b_1)}. \tag{31}$$

With respect to the definition of the new parameters $\delta \to \delta - \alpha\varepsilon$, and $f \to f - \delta\varepsilon + 0.5\alpha\varepsilon^2$, these solutions are (separately) invariant as they represent $\tilde{u}_2$ as a linear function of $y$ and $\tilde{h}_1$ as a quadratic function of $y$. This means $\mathcal{G}_2$ consists of parameter invariant transformations of $X_4$.

The action of $\mathcal{G}_4$ changes the solutions as follows:

$$\begin{aligned} \tilde{u}_1 &= \frac{a_1 x + c_1 e^{\varepsilon}}{a_1 t + b_1}, \quad \tilde{u}_2 = (\alpha y + \delta e^{\varepsilon})g(t), \\ \tilde{h}_1 &= \frac{d_1\left(\alpha\frac{y^2}{2} + \delta e^{\varepsilon} y + f e^{2\varepsilon}\right)}{\beta_2(a_1 t + b_1)}, \quad \text{and} \quad \tilde{h}_2 = K e^{2\varepsilon} e^{-\alpha\int g(t)\,dt}. \end{aligned} \tag{32}$$

It is clear from (32) that $\mathcal{G}_4$ changes individual solutions from $X_4$. However, the transformed solutions are again in $X_4$ as the number of parameters determining solutions do not change. In other words, action of $\mathcal{G}_4$ applied to solutions from $X_4$ qualitatively (in the form) produce similar solutions.

The action of $\mathcal{G}_5$ changes the solutions as follows:

$$\begin{aligned} \tilde{u}_1 &= \frac{a_1 x + c_1}{a_1 t + b_1 e^{\varepsilon}}, \quad \tilde{u}_2 = e^{-\varepsilon}(\alpha y + \delta)g(e^{-\varepsilon}t), \\ \tilde{h}_1 &= \frac{d_1 e^{-\varepsilon}\left(\frac{\alpha y^2}{2} + \delta y + f\right)}{\beta_2(a_1 t + b_1 e^{\varepsilon})}, \quad \text{and} \quad \tilde{h}_2 = K e^{-2\varepsilon} e^{-\alpha e^{-\varepsilon}\int g(e^{-\varepsilon}t)\,dt}. \end{aligned} \tag{33}$$

In (33), the action of $\mathcal{G}_5$ on $u_1$ and $h_1$ does not change them significantly. But the number of parameters spanning $\tilde{u}_2$ and $\tilde{h}_2$ has increased as compared to the number of parameters used in $u_2$ and $h_2$. In particular, $(\tilde{u}_1, \tilde{h}_1, \tilde{u}_2, \tilde{h}_2) \notin X_4$ and hence we have new families of solutions of the system (6)–(7).

The actions of $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ on $X_5$ can be analyzed easily as above. It is more interesting to analyze actions of $\mathcal{G}_4$ and $\mathcal{G}_5$ on $X_5$. The action of $\mathcal{G}_4$ is as follows:

$$\begin{aligned} \tilde{u}_1 &= e^{\varepsilon}\left(\frac{a_1 e^{-\varepsilon}x + c_1}{a_1 t + b_1} - \int \gamma(t)\,dt + \frac{a_1}{a_1 t + b_1}\int\left(\int \gamma(t)\,dt\right)dt\right), \quad \tilde{u}_2 = (\alpha y + \delta e^{\varepsilon})g(t), \\ \tilde{h}_1 &= \frac{d_1\left(\alpha\frac{y^2}{2} + \delta e^{\varepsilon} y + f e^{2\varepsilon}\right)}{\beta_2(a_1 t + b_1)}, \quad \text{and} \quad \tilde{h}_2 = e^{2\varepsilon}(p e^{-\varepsilon}x + q)e^{-\alpha\int g(t)\,dt}. \end{aligned} \tag{34}$$

And the action of $\mathcal{G}_5$ is:

$$\begin{aligned} \tilde{u}_1 &= \frac{a_1 x + c_1}{a_1 t + b_1 e^{\varepsilon}} - e^{-2\varepsilon}\int \gamma(e^{-\varepsilon}t)\,dt + \frac{a_1 e^{-2\varepsilon}}{a_1 t + b_1 e^{\varepsilon}}\int\left(\int \gamma(e^{-\varepsilon}t)\,dt\right)dt, \quad \tilde{u}_2 = e^{-\varepsilon}(\alpha y + \delta)g(e^{-\varepsilon}t), \\ \tilde{h}_1 &= \frac{e^{-\varepsilon}d_1\left(\frac{\alpha y^2}{2} + \delta y + f\right)}{\beta_2(a_1 t + b_1 e^{\varepsilon})}, \quad \text{and} \quad \tilde{h}_2 = e^{-2\varepsilon}(px + q)e^{-\alpha e^{-\varepsilon}\int g(e^{-\varepsilon}t)\,dt}. \end{aligned} \tag{35}$$

The presence of the term of the form $-\int \gamma(t)\,dt + \frac{a_1}{a_1 t + b_1}\int\left(\int \gamma(t)\,dt\right)dt$ in $u_1$ makes the situation much more interesting. Under the action of these transformations, the structure of the third term of $\tilde{u}_1$ is different than the structure of the third term of $u_1$ and consequently $\mathcal{G}_4(X_5) \not\subset X_5$ and $\mathcal{G}_5(X_5) \not\subset X_5$. This means that we are able to construct solutions which are not in $X_5$, in other words, the action of $\mathcal{G}_4$ and $\mathcal{G}_5$ on $X_5$ generate new solutions. This justifies our analysis of action of one parameter subgroups on the solutions. Finally, we note that this observation is possible because of $K$ being a non-constant function. Hence, the nature of the function $K$ plays a dominant role in the final form of the solutions.

### *6.2. Relative non-invariance*

The solutions produced in Section 6.1 from the actions of the symmetry transformations appear qualitatively similar in the form as those before transformations. However, these transformed solutions are quantitatively different and have different physical significance than the un-transformed solutions. The main reason for this is that the transformed solutions do not transform $u_1$ and $u_2$ to $\tilde{u}_1$ and $\tilde{u}_2$ by the same quantity but the group actions are applied to their arguments. In other words, these mappings (group actions) are not on the total values but on the arguments. The same is true for the flow depths $\tilde{h}_1$ and $\tilde{h}_2$.

The key objects of interest in the mixture flows are not the velocities of the solid and the fluid phases separately, but the relative velocity between the phases which is a dominant dynamical quantity. For example, the drag and the virtual mass forces are induced by the non-zero relative velocity and non-zero relative acceleration [18], but the values of the phase velocities and accelerations separately are not relevant. So, from the mechanical point of view it is not the value of the phase velocities but their difference that is important. Although we saw that some symmetry transformations qualitatively map solutions to the same class of solutions, with re-defined parameters which are combinations of the basic parameters in the functions before transformation and associated group parameters, these transformations can be realized as non-invariance transformations with respect to the prospective and importance of the underlying physics of the mixture material. This develops into a concept of relative non-invariance.

**Definition.** A mapping $G : S \to S$ induced by a symmetry transformation is said to be a relative invariance if for all solutions $(u_1, h_1, u_2, h_2)$ in $S$, $G(u_1) - G(u_2) = u_1 - u_2$ and $G(h_1) - G(h_2) = h_1 - h_2$. A mapping $G : S \to S$ which is not a relative invariance will be called a relative non-invariance.

Now, using transformations (32), we demonstrate the relative non-invariance of the transformed solutions for the action of $\mathcal{G}_4$. With a little algebra, we obtain:

$$\tilde{u}_1 - \tilde{u}_2 = (u_1 - u_2) + (e^{\varepsilon} - 1)\left[\frac{c_1}{a_1 t + b_1} - \delta g(t)\right]. \tag{36}$$

This shows that the relative velocity of the transformed solutions is invariant to the given solutions if and only if either $\varepsilon = 0$, or $g(t) = c_1/\delta(a_1 t + b_1)$. However, neither $\varepsilon = 0$ nor, $g(t) = c_1/\delta(a_1 t + b_1)$, as $g(t)$ can be different in general. So, the transformation (32) is not invariant in general with respect to the relative velocity between the phases.

Defining invariance with their relative values also applies to the flow depths. It is even more apparent because $h_1 - h_2$ measures the difference in the flow heights between the phases, thus the distance. The transformed relative height reads:

$$\tilde{h}_1 - \tilde{h}_2 = (h_1 - h_2) + \frac{d_1\left[(e^{\varepsilon} - 1)\,\delta y + \left(e^{2\varepsilon} - 1\right) f\right]}{\beta_2\,(a_1 t + b_1)} - \left(e^{2\varepsilon} - 1\right) K e^{-\alpha \int g(t)dt}. \tag{37}$$

This shows that as, in general, $\varepsilon \neq 0$, the considered transformation is a relative non-invariance. It has even more dynamical implications because it is directly associated with the solid (or, fluid) volume fraction in the

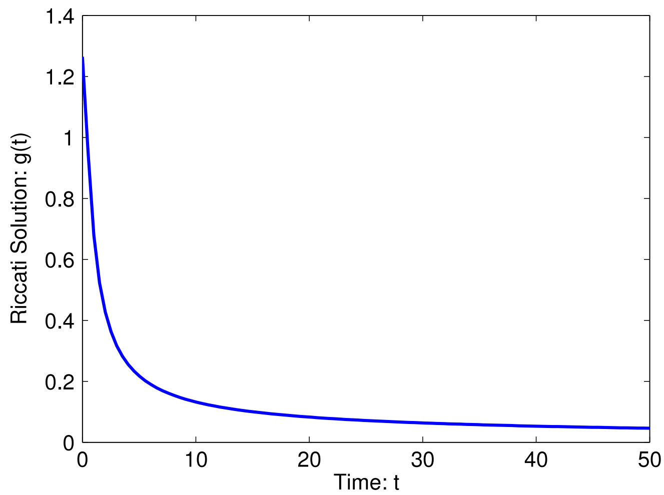


**Fig. 1.** General solution of the Riccati equation (38).

mixture and its evolution. As the measure of the relative flow height $(h_1 - h_2)/h = 2\alpha_s - 1$, its importance is obvious because, volume fraction is one of the most prominent dynamical quantities in the mixture flows. However, we note that these transformations would be invariant for the single-phase flows but apparently not for the multi-phase flows with respect to the relevance of the difference in the corresponding dynamical quantities under consideration.

## 7. Discussions on results

### 7.1. Behavior of the Riccati solution

We observe that solutions $g(t)$ of the Riccati equation (18) plays a crucial role in the time evolution of the solution of the physical system (6)–(7). It is possible to write solutions of the Riccati equation in closed form in terms of the complex composition of Bessel's functions of first and second kind of zeroth, first and second orders. Further, complicated time behavior of the solutions appear with the integrand of $g(t)$, its exponent and recursions in the above solutions with $\gamma(t)$. Let us consider a simplification of the Riccati's equation (18) of the form

$$(a+t)g'(t) + c(a+t)g(t)^2 + b = 0. \tag{38}$$

For suitably arranged material constants $a, b, c$ the general solution of (38) is given by

$$g(t) = \left[\lambda\left(-bcI_1/T - 0.5bc(I_0 + I_2)\right) + 0.5bc(K_0 + 0.5K_1/T - K_2)\right] / \left[b(0.5\lambda T I_1 - 0.5TK_1)\right], \tag{39}$$

where, $\lambda$ is a constant of integration, and $T = 2\sqrt{-bc(a+t)}$ is the argument in the Bessel's functions. Similarly, $I_n = I_n(T)$ and $K_n = K_n(T)$ are the modified Bessel's functions of the first and second kinds respectively, of orders $n = 0, 1, 2$. Fig. 1 shows a Riccati solution for the parameter choice $(a, b, c, \lambda) = (1.0, 0.85, 0.25, 1.1)$. The figure reveals a hyperbolic-type function of $t$, which is more general than the simple hyperbolic solution of (12).

### 7.2. Spatial evolution of the solid and fluid phases

The exact solutions developed above can be analyzed for their space or time (or, both) evolutions of the solid and fluid phases. For better understanding, here, we analyze the results for space, and time

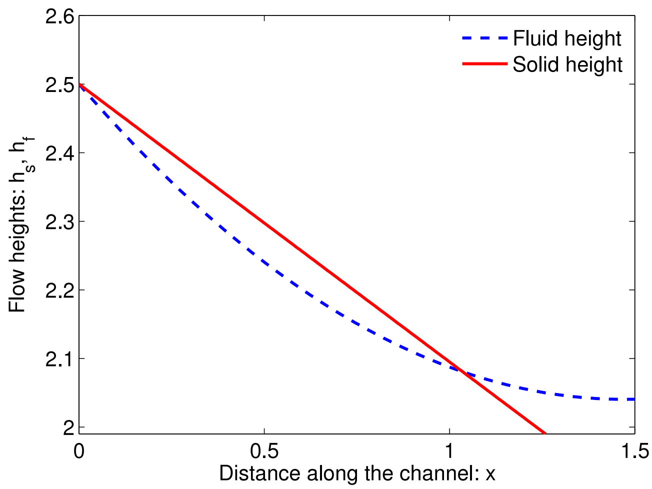


**Fig. 2.** Spatial evolution of the solid and fluid phases as the mixture slides down an inclined channel. The mixture is released from a silo gate from the top left.

evolutions separately. Fig. 2 shows the spatial evolution of the solid and the fluid phases as a mixture material released from a silo gate flows down an inclined slope. The figure corresponds to the solutions $h_f = a(\alpha y^2/2 + dy + f)/(\beta_2(t+b))$, and $h_s = (py+q)\exp\left(-\alpha \int g(t)dt\right)$ immediately after the flow release, with parameter values $a = 0.25, \alpha = 0.85, d = -1.25, f = 5.0, \beta_2 = 0.5, b = 1.0, p = -0.405$, and $q = 2.5$. We observe that although the solid flow depth decreases linearly along the downslope the decrease of the fluid phase is quadratic. In the vicinity of the release gate the fluid height decreases quickly than the solid height, however, farther downstream the fluid height decreases only slowly whereas the solid height decreases continuously. The fluid evolution shows more realistic solution than the solid. On the other hand, generally the solid (front) shows a tapering behavior whilst the fluid phase deformation is more non-linear, or quadratic. These are observable phenomena in mixture flows down a slope [18,23].

### *7.3. Time evolution of the solid and fluid phases*

Next, we fix a downslope position (say, $x_p$) and analyze the behavior of solution in time at that position. The structure of the solution $g(t)$ of the Riccati function indicates that $exp\left(-\alpha \int g(t)dt\right)$ can be approximated by an exponentially decaying function in time. This means that, at a given position along the slope, both the solid and fluid flow depth decrease inversely with time, but differently, because the solid and fluid material parameters are different, so does their dynamics. There are two possible scenarios. First, consider a mixture flowing down and consider the situation when the flow head (surge) has just passed away the reference point $x_p$. Then, as the debris material stretches and thins in the back and tail side of the flow body, the time evolution of the flow depths for both the solid and fluid decrease inversely in time [3,5,38] at position $x_p$. Second, consider a typical situation such that the flow propagates down slope but decelerates slowly. The exact solutions constructed above can explain the flow dynamics for both the flow depth and velocities. The solutions for velocities also vary inversely with time. Since $x_p$ is a position behind the surge, for decelerating flow both the flow depths and flow velocities decrease. This is a plausible scenario before the flow transits to the deposition. These solutions clearly show the decelerating flow and decrease in both flow heights. Here, we do not aim for quantitative analysis. This can however be done with, say, laboratory data that constrain the numerical values for the model parameters. These exact solutions presented above might be extended to construct more realistic and general non-linear solutions for both the solid and fluid phases.

## 8. Summary

In this paper, we investigated a two-phase mixture mass flow model by constructing several families of analytical solutions with their physical significance. Solutions are derived with splitting the equation and applying the method of separation of variables and characteristics. Splitting significantly helped to reduce the given problem into relatively easier ones, e.g., transforming the non-linear PDEs to linear and quasi-linear PDEs, whose solutions provide insights into the solution of the full problem. This led to the Burgers, and Riccati-type equations. Corresponding analytical solutions are obtained with increased complexities for the phase velocities and the phase heights as functions of space and time. Particularly, the velocity solutions can be represented as a two-parameter family of functions.

There are important implications of the analytical solutions for the phase heights, and phase velocities which are functions of both space and time. Velocity and flow depth solutions are up to quadratic in the space variable, but appear to be very complex combination of the Bessel's functions of the first and second kinds, and the solution of the general Riccati-type equation for the time evolution, where the Riccati-equation emerges from the momentum equations of the considered model. We analyzed a Riccati-solution, which played a crucial role in the representation of the velocity and flow depth solutions, in detail. Solutions indicate that change in the flow depth of a phase influences the velocity of the other phase. This implies a strong and direct coupling between the phases in the mixture flow. Particularly important observation is that these solutions are directly coupled through the physical model parameters of the solid and the fluid phases, and the Riccati-solution.

Moreover, by applying the Lie group action, we transformed the newly obtained analytical solutions to generate further solutions and analyzed their possible invariance. In doing so, we presented new perspectives and definitions of an invariance transformation associated to the underlying physics relevant to multi-phase flows, namely, the relative velocity and relative flow depths between the phases. We found that the transformed solutions are quantitatively different and result in the different physical significance than the un-transformed solutions. With respect to the importance of the underlying physics of the mixture material these transformations appear to be relatively non-invariant. We realized that importance in the mixture flows is not the velocities of the solid and the fluid phases separately but, the relative velocity and flow depths between the phases which are the dominant dynamical quantities in the mixture flows. So, from the mechanical point of view it is not the value of the phase velocities but their difference that is important. This led to a concept of the relative non-invariance transformation. We demonstrated the relative non-invariance of the transformed solutions. Nevertheless, with respect to the definition of the new set of parameters, all the solutions thus obtained under the action of the Lie group transformation remain structurally (qualitatively) and individually invariant.

Finally, we presented detailed analysis and discussion on the time and spatial evolutions of the analytical solutions for solid and fluid phase velocities and the flow depths. The obtained analytical solutions are in line with the physics of two-phase mass flows down a slope. So, the exact solutions presented here might be extended to construct more realistic and general non-linear solutions for both the solid and fluid phases.

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