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A novel description of fluid flow in porous and debris materials

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ABSTRACT

Based on a generalized two-phase mass flow model (Pudasaini, 2012) as a mixture of solid particles and interstitial fluid, here, we derive a novel dynamical model equation for sub-diffusive and sub-advective fluid flow in general porous media and debris material in which the solid matrix is stationary. We construct some exact analytical solutions to the new model. The complete exact solutions are derived for the full sub-diffusive fluid flows. Solutions for the classical linear diffusion and the new sub-diffusion with quadratic fluxes are compared, and the similarities and differences are discussed. We show that the solution to sub-diffusive fluid flow in porous and debris material is fundamentally different from the diffusive fluid flow. In the sub-diffusive process, the fluid diffuses slowly in time, and thus, the flow (substance) is less spread. Furthermore, we construct some analytical solutions for the full sub-diffusion and sub-advection equation by transforming it into classical diffusion and advection structure. High resolution numerical solutions are presented for the full sub-diffusion and sub-advection model, which is then compared with the solution of the classical diffusion and advection model. Solutions to the sub-diffusion and sub-advection model reveal very special flow behavior, namely, the evolution of forward advecting frontal bore head followed by a gradually thinning tail that stretches to the original rear position of the fluid. However, for the classical diffusion–advection model, the fluid simply advects and diffuses. Moreover, the full sub-diffusion and sub-advection model solutions are presented both for the linear and quadratic drags, which show that the generalized drag plays an important role in generating special form and propagation speed of the sub-diffusion–advection waves. We also show that the long time solution to sub-diffusive and sub-advective fluid flow through porous media is largely independent of the initial fluid profile. These exact, analytical and numerical solutions reveal many essential physical phenomena, and thus may find applications in modeling and simulation of environmental, engineering and industrial fluid flows through general porous media and debris materials.

1. Introduction

Macroscopic averaged equations describing fluid flow through porous media are of great practical and theoretical interest in science and technology including oil exploration and environment related problems (Darcy, 1856; Richards, 1931; Bear, 1972; De Marsily, 1986; Durlofsky and Brady, 1987; Dagan, 1989). There are several geophysical and industrial applications of such fluid flows. This includes the flow of liquid or gas through soil and rock (e.g., shell oil extraction), clay, gravel and sand, or through sponge and foam. From geophysical and engineering prospectives, the fluid flows through porous and debris media are important aspects as it is coupled with the stability of the slope, the subsurface hydrology, and the transportation of chemical substances in porous landscape. Proper understanding of fluid flows in debris material and porous landscape, and in general through porous media, is an important aspect in industrial applications, geotechnical engineering, engineering geology, subsurface hydrology and natural hazard related phenomena (Muskat, 1937; Barenblatt, 1952; Bear, 1972; Whitaker, 1986; Boon and Lutsko, 2007; Vazquez, 2007). Better and reliable understanding of slope stability analysis, landslide initiation, debris and avalanche deposition morphologies, including seepage of fluid through relatively stationary porous matrix and consolidation, require more accurate and advanced knowledge of fluid flows in porous materials. Understanding the dynamics of fluid flows in porous landscape may help to develop early warning strategies in potentially huge and catastrophic failure of landslides, reservoir dams and embankments in geo-disaster-prone areas (Genevois and Ghirotti, 2005; Pudasaini and Hutter, 2007; Khattri, 2014; Miao et al., 2014; Pudasaini, 2014), and deposition processes of subsequent mass flows (Zhang et al., 2011; Kuo et al., 2011; Mergili et al., 2012; Tai and Kuo, 2012; Fischer, 2013; Wang et al., 2013; Zhang and Yin, 2013; Yang et al., 2015). Here, the terms porous landscape, debris material and porous media are used as synonym, because in all these materials we assume that the fluid passes through the relatively stationary solid skeleton, or matrix of granular particles.

Classically, the flow of a fluid through a homogeneous porous medium is described by the porous medium equation. The model is derived by using the continuity equation for the flow of ideal fluid through porous medium, the Darcy law relating fluid pressure gradient to the mean velocity, and by assuming a state equation for ideal fluid in

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which pressure is an explicit exponent (≥ 1) function of the fluid (gas) density. The resulting porous media equation is similar in form to the classical diffusion equation, except that, in the porous medium equation, the diffusive flux is non-linear with exponent ≥ 2 (Bear, 1972; Smyth and Hill, 1988; Vazquez, 2007) which is ultimately responsible for the slow diffusion process. Exact solutions exist for the porous medium equation (Barenblatt, 1952, 1953; Evans, 2010; Boon and Lutsko, 2007) in similarity form. Unlike the classical diffusion equation, the solution for porous medium equation is bounded by finite propagation speed of the flow fronts, and a compact support (Daskalopoulos, 2009), which means the solution is contained within a fixed domain.

Based on the generalized two-phase mass flow model (Pudasaini, 2012) this paper presents a novel sub-advection and sub-diffusion model equation and its analytical solutions for fluid flows through inclined debris materials and porous landscapes with stationary solid matrix. In these scenarios, we show that a number of important physical phenomena are rather governed by the nonlinear advection and diffusion processes that are associated with exact description of physical model parameters. Our results reveal that solutions to the sub-diffusive fluid flow in porous media and porous landscape is fundamentally different from the classical diffusive fluid flow or diffusion of heat, tracer particles and pollutant in fluid. Reduced models and exact solutions are presented for the sub-diffusive fluid flow. We outline some possible and systematic ways to construct analytical solutions for our new full sub-diffusion and sub-advection equation (also see, Boon and Lutsko, 2007). This includes the transformation of the model (in the form) to the convenient classical diffusion–advection equation for which we have constructed advanced analytical solutions by using the Bring ultraradical (Bring, 1864) and higher-order hypergeometric function. Furthermore, separation of variables leads to special ordinary differential equations in the form of Lienard and Abel canonical equations, that may provide other set of exact solutions (Lienard, 1928). We have presented numerical solution to such model that provides insight into the intrinsic nature of fluid flow in porous media.

New analytical solutions for fluid flows through debris material and porous landscape are then compared with numerical simulations to measure the performance of the numerical method for their further use in relevant fluid flows. The widely used high-resolution, shock-capturing Total Variation Diminishing Non-oscillatory Central (TVD-NOC) scheme (Nessyahu and Tadmor, 1990; Tai et al., 2002; Pudasaini, 2011) is implemented to solve the model equations numerically. For alternative numerical methods including finite volume, discrete element methods, spring-deformable-block model, we refer to Crosta et al. (2003), Mangeney-Castelnau et al. (2003), Denlinger and Iverson (2004), Pirulli (2009), Teufelsbauer et al. (2011) and Yang et al. (2015). Numerical results are presented for the full sub-diffusion and sub-advection model, which is then compared with the solution of the classical diffusion and advection model. Special features associated with the new model, and as revealed by the exact, analytical, and numerical solutions, are discussed in detail. Moreover, the full sub-diffusion and sub-advection model solutions are presented both for the linear and quadratic drags. The long time solutions are analyzed for different initial fluid profiles.

On the one hand, exact, analytical, and numerical solutions disclose many new and essential physics, and thus, may find applications in environmental, engineering and industrial fluid flows through general porous media, natural slopes, embankments (e.g., of hydro-electric power reservoirs and dams), and debris materials. While on the other hand, analytical and exact solutions to simplified cases of nonlinear model equations are necessary to calibrate numerical solution methods (Pudasaini, 2011). The reduced and problem-specific solutions provide important insights into the full behavior of the complex two-phase system, mainly the flow of fluid through the porous media. Broadly speaking, these results can further be applied to the problems related to hydrogeology, and environmental pollution remediation.

2. The two-phase mixture model

In order to develop a new sub-diffusion and sub-advection model for fluid flow through a porous media, we consider the general two-phase mass flow model (Pudasaini, 2012) that describes the dynamics of a real two-phase debris flow as a mixture of the solid particles and the interstitial fluid. The model is developed within the framework of continuum mechanics. For more on multi-phase and other relevant flows, we refer to Richardson and Zaki (1954), Anderson and Jackson (1967), Drew (1983), Ishii and Hibiki (2006), and Kolev (2007). The two phases are characterized by distinct material properties: the fluid phase is characterized by its true density ρ_f, viscosity η_f, and isotropic stress distribution, whereas the solid phase is characterized by its material density ρ_s, internal and basal friction angles, ϕ and δ, respectively, and an anisotropic stress distribution, K (lateral earth pressure coefficient). These characterizations and the presence of relative motion between these phases result in two different mass and momentum balance equations for the solid and the fluid phases, respectively. Let u_s, u_f and α_s, α_f $(=1\text{-}\alpha_s)$ denote the velocities, and volume fractions for the solid and the fluid constituents, denoted by the suffices s and f, respectively. The general two-phase debris flow model reduced to one-dimensional flows down a slope are described by the following set of non-linear partial differential equations (Pudasaini, 2012, 2014; Pudasaini and Miller, 2012; Pudasaini and Krautblatter, 2014):

$$\frac{\partial}{\partial t}(\alpha_s h) + \frac{\partial}{\partial x}(\alpha_s h u_s) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\alpha_f h) + \frac{\partial}{\partial x}(\alpha_f h u_f) = 0, \tag{2}$$

$$\frac{\partial}{\partial t}\left[\alpha_s h(u_s - \gamma\mathcal{C}(u_f - u_s))\right] + \frac{\partial}{\partial x}\left[a_s h\left(u_s^2 - \gamma\mathcal{C}\left(u_f^2 - u_s^2\right) + \frac{\beta_s h}{2}\right)\right] = hS_s, \tag{3}$$

$$\frac{\partial}{\partial t}\left[\alpha_f h\left(u_f + \frac{\alpha_s}{\alpha_f}\mathcal{C}(u_f - u_s)\right)\right] + \frac{\partial}{\partial x}\left[a_f h\left(u_f^2 + \frac{\alpha_s}{\alpha_f}\mathcal{C}\left(u_f^2 - u_s^2\right) + \frac{\beta_f h}{2}\right)\right] = hS_f. \tag{4}$$

Eqs. (1) and (2) are the depth-averaged mass balances, and Eqs. (3) and (4) are the depth-averaged momentum balance equations for the solid and the fluid phases, respectively.

The force/source terms in the momentum equation for the solid-phase (Eq. (3)) is

$$S_s = \alpha_s\left[g^x - \frac{u_s}{|u_s|}\tan\delta p_{b_s} - \varepsilon p_{b_s}\frac{\partial b}{\partial x}\right] - \varepsilon\alpha_s\gamma p_{b_f}\left[\frac{\partial h}{\partial x} + \frac{\partial b}{\partial x}\right] + C_{DG}(u_f - u_s)|u_f - u_s|^{j-1}. \tag{5}$$

Similarly, the force/source term for the fluid-phase (Eq. (4)) is

$$S_f = \alpha_f\left[g^x - \varepsilon\left[\frac{1}{2}p_{bf}\frac{h}{\alpha_f}\frac{\partial\alpha_s}{\partial x} + p_{bf}\frac{\partial b}{\partial x} - \frac{1}{\alpha_f N_R}\left\{2\frac{\partial^2 u_f}{\partial x^2} - \frac{\chi u_f}{\varepsilon^2 h^2}\right\} + \frac{1}{\alpha_f N_{R_\mathcal{A}}}\left\{2\frac{\partial}{\partial_x}\left(\frac{\partial\alpha_s}{\partial x}(u_f - u_s)\right)\right\} - \frac{\xi\alpha_s(u_f - u_s)}{\varepsilon^2\alpha_f N_{R_\mathcal{A}} h^2}\right]\right] - \frac{1}{\gamma}C_{DG}(u_f - u_s)|u_f - u_s|^{j-1}. \tag{6}$$

In Eqs. (5) and (6), $j = 1$ or, 2 correspond to the linear, or quadratic drags. The other parameters are

$$\begin{gathered}\beta_s = \varepsilon K p_{b_s}, \quad \beta_f = \varepsilon p_{b_f}, \quad p_{b_f} = -g^z, \quad p_{b_s} = (1-\gamma)p_{b_f},\\ C_{DG} = \frac{\alpha_s\alpha_f(1-\gamma)}{\left[\varepsilon\,\mathcal{U}_T\{\mathcal{P}\mathcal{F}(Re_p) + (1-\mathcal{P})\mathcal{G}(Re_p)\}\right]^j}, \quad \mathcal{F} = \frac{\gamma}{180}\left(\frac{\alpha_f}{\alpha_s}\right)^3 Re_p, \quad \mathcal{G} = \alpha_f^{M(Re_p)-1},\\ \gamma = \frac{\rho_f}{\rho_s}, \quad Re_p = \frac{\rho_f d\,\mathcal{U}_T}{\eta_f}, N_R = \frac{\sqrt{gL}H\rho_f}{\alpha_f\eta_f}, \quad N_{R_\mathcal{A}} = \frac{\sqrt{gL}H\rho_f}{\mathcal{A}\eta_f}, \quad \alpha_f = 1-\alpha_s.\end{gathered} \tag{7}$$

In the above equations, t is the time, h is the flow depth (or, porous material height). p_{b_f} and p_{b_s} are associated with the effective fluid and solid pressures. x and z are coordinates along the flow directions, and

g^x and g^z are the components of gravitational acceleration along the x and z directions, respectively (Fig. 1). L and H are the typical length and depth of the flow, $\varepsilon = H/L$ is the aspect ratio, and $\mu = \tan\delta$ is the basal friction coefficient. The earth pressure coefficient, K is a function of δ, and ϕ, basal and internal friction angles of the solid particles, C_{DG} is the generalized drag coefficient. $\mathcal{U}_T$ is the terminal velocity of a particle (Richardson and Zaki, 1954; Pitman and Le, 2005) and $\mathcal{P} \in [0, 1]$ is a parameter which combines the solid-like ($\mathcal{G}$) and fluid-like ($\mathcal{F}$) drag contributions to flow resistance. γ is the density ratio, $\mathcal{C}$ is the virtual mass coefficient, M is a function of the particle Reynolds number (Re_p) (Richardson and Zaki, 1954; Pitman and Le, 2005), χ includes vertical shearing of fluid velocity, and ξ takes into account the different distributions of α_s. $\mathcal{A}$ is the mobility of the fluid at the interface, and N_R and N_{R_A} are quasi-Reynolds numbers associated with the classical Newtonian, and enhanced non-Newtonian fluid viscous stresses, respectively (Pudasaini, 2012). The impermeable slope topography is represented by $b = b(x)$. Although there seems to be lots of parameters in the list (Eq. (7)), the parameters are well defined and well constrained from laboratory, or field data. Furthermore, most of the parameters are derived from a very few basic parameters (Pudasaini, 2012).

Some important aspects of the model

In Eqs. (3)–(4), left hand sides are inertial terms which include the lateral pressures (associated with β_s and β_f) and the virtual mass coefficient, $\mathcal{C}$. The source in the solid momentum (Eq. (5)) has different contributions: the first square bracket on the right hand side is associated with gravity, the Coulomb friction and the slope gradient; the second square bracket is associated with the buoyancy force; and the last term is associated with the generalized drag contribution (C_{DG}). The source term for the fluid-momentum (Eq. (6)) also has multiple contributions to force. The first three terms on the right hand side emerge from the gravity load applied to the fluid phase (first term), the fluid pressure at the bed (second term) and the topographic slope (third term). The fourth group of terms associated with N_R emerges from the viscous force contribution of the fluid phase. The fifth group of terms associated with N_{R_A} is the non-Newtonian viscous contribution which occurs because, viscous shear stress is enhanced by the solid-volume–fraction gradient.

As discussed in detail in Pudasaini (2012), Eqs. (1)–(4) unifies the three pioneering theories in geophysical mass flows, the dry granular avalanche model of Savage and Hutter (1989), the debris-flow model of Iverson (1997) and Iverson and Denlinger (2001), and the two-fluid debris-flow model of Pitman and Le (2005), and result in a new, generalized two-phase debris-flow model. As special cases of the new general debris flow model, one recovers these relatively simple models for debris flows and avalanches (Pudasaini, 2012).

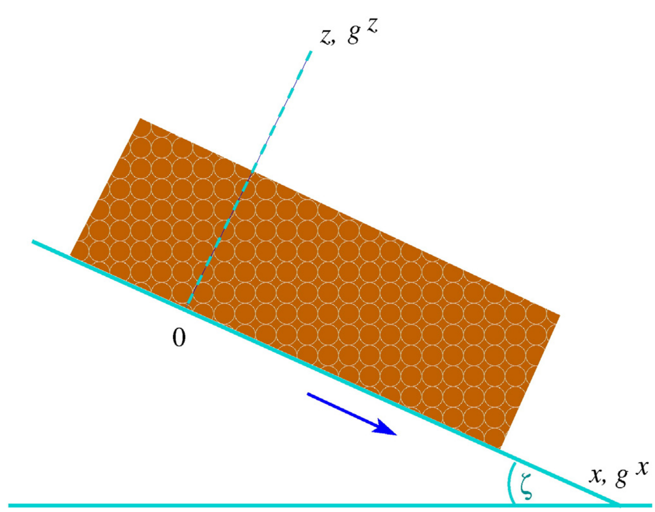

Fig. 1. Sketch of the coordinates and geometry of an inclined porous landscape, or debris material in which the solid skeleton remains stationary and undeformed, and the fluid flows through it. The arrow indicates fluid flow direction. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

3. Derivation of a new generalized porous media equation

We consider the two-phase mixture flow (Eqs. (1)–(6)). Here, we are particularly interested to develop a new and simple model for the fluid flow through a relatively stationary porous debris material. The main idea is as follows: Since the real two-phase debris mass flow consists of the matrix of solid particles and interstitial fluid, both the solid- and fluid-phases may move and deform. The advantage of a real two-phase mixture model is that either we can consider the motion and deformation of both the solid- and the fluid-phases, or one phase may move and deform while the other phase may remain stationary (not moving, not deforming). Either of these conditions may prevail in nature. However, in certain situations, the motion and deformation of the solid matrix, thus the solid-phase, can be negligible as compared to the flow of fluid through a stationary solid matrix. Examples include the fluid flow through the solid matrix before the failure of a landmass resulting in a debris motion, during the deposition process of the debris material in the run-out zone where the fluid may seep through the relatively stationary solid matrix. Similar phenomena can be observed in fluid flows through embankments of reservoir and water filtration plants. So, motivated by these applications, here we only consider the flow of fluid (that can be fast or relatively slow) but the motion and deformation of the solid-phase is negligible.

Therefore, for the purpose of developing a model for fluid flow through a stationary solid matrix it is legitimate to assume that the solid deformation and motion is negligible as compared to the fluid in the system (Eqs. (1)–(6)). Here, the term 'debris' does not mean the 'debris flow', it rather stands for the porous debris material or landscape in which the solid matrix does not move and deform but the fluid flows through the stationary solid matrix. So, $u_s \ll u_f$ is physically justified. Furthermore, as in Ancey (2007), Takahashi (2007), and Pudasaini (2011), we assume a gentle slope and that the flow is non-inertial. Then, Eqs. (2) and (4) (including the hydraulic pressure gradient term associated with β_f) with source term (Eq. (6)), reduce to the simple mass and momentum balances for the fluid flow through the stationary porous landscape and debris material in the down-slope (x) direction:

$$\frac{\partial}{\partial t}(\alpha_f h) + \frac{\partial}{\partial x}(\alpha_f h u_f) = 0, \tag{8}$$

$$\frac{2}{N_R}\frac{\partial^2 u_f}{\partial x^2} - \left(\frac{2}{N_{R_A}}\frac{\partial \alpha_s}{\partial x}\right)\frac{\partial u_f}{\partial x} - \left(\frac{1}{\gamma}C_{DG}u_f^{j-1} + \frac{2}{N_{R_A}}\frac{\partial^2 \alpha_s}{\partial x^2}\right)u_f + \alpha_f\left(\sin\zeta - \cos\zeta\frac{\partial h}{\partial x}\right) = 0. \tag{9}$$

As mentioned above, h is the debris (or porous) material height, ζ is the slope angle, and α_s, α_f are the solid and fluid volume fractions, respectively.

In this section, we derive a novel dynamical model equation for sub-diffusive and sub-advective flow of fluid in porous and debris materials. In Eq. (9), for simplicity, if we set $N_R, N_{R_A} \to \infty$, this results in:

$$u_f^j = \frac{\gamma}{C_{DG}}\cos\zeta\ \alpha_f\left(\tan\zeta - \frac{\partial h}{\partial x}\right), \tag{10}$$

which is the generalized Darcy expression for the fluid velocity (Darcy, 1856; Pudasaini, 2012). Darcy flow is characterized by a relatively simple expression $u_f = -K\partial h/\partial x$, where K is hydraulic conductivity. For simplicity, assume the linear drag, $j = 1$, otherwise, $j = 2$ can also be considered, see Section 5, and Fig. 6 later. Now, we consider the down-slope fluid mass balance Eq. (8). Substituting u_f from Eq. (10), Eq. (8) takes the form

$$\frac{\partial}{\partial t}(\alpha_f h) - \frac{\partial}{\partial x}\left[\frac{\gamma\cos\zeta}{C_{DG}}\alpha_f(\alpha_f h)\frac{\partial h}{\partial x}\right] + \frac{\partial}{\partial x}\left[\frac{\gamma\sin\zeta}{C_{DG}}\alpha_f(\alpha_f h)\right] = 0. \tag{11}$$

In a diffusion process, the gradient of the diffusing substance generally decreases. As we will see later, for the present analysis, the important and relevant variable is $\alpha_f h$ rather than α_f. So, with respect to the underlying diffusion process, we may assume that $\partial\alpha_f/\partial x$ is small. Otherwise, there would be an extra source term in the emerging model equation. We know that $\gamma<1$, and $\cos\zeta<1$. For fluid passing through the solid matrix, $1/C_{DG}<1$ (Pudasaini, 2012). This implies that $(\gamma\cos\zeta/C_{CD})\partial\alpha_f/\partial x \ll 1$, and thus can be neglected. So, the terms within the first square bracket in Eq. (11) can be approximated as

$$\frac{\gamma\cos\zeta}{C_{DG}}\alpha_f(\alpha_f h)\frac{\partial h}{\partial x} = (\alpha_f h)\left[\frac{\gamma\cos\zeta}{C_{DG}}\frac{\partial(\alpha_f h)}{\partial x} - h\frac{\gamma\cos\zeta}{C_{DG}}\frac{\partial\alpha_f}{\partial x}\right] \approx (\alpha_f h)\left[\frac{\gamma\cos\zeta}{C_{DG}}\frac{\partial(\alpha_f h)}{\partial x}\right]. \tag{12}$$

Since $\alpha_f^2 << 1$, $\gamma<1$, $\sin\zeta<1$ and $1/C_{DG}<1$ we obtain the following approximation for the third term in Eq. (11):

$$\frac{\partial}{\partial x}\left[\left(\frac{\gamma\sin\zeta}{C_{DG}h}\right)(\alpha_f h)^2\right] = \left(\frac{\gamma\sin\zeta}{C_{DG}h}\right)\frac{\partial(\alpha_f h)^2}{\partial x} - \alpha_f^2\left(\frac{\gamma\sin\zeta}{C_{DG}}\right)\frac{\partial h}{\partial x} \approx \left(\frac{\gamma\sin\zeta}{C_{DG}h}\right)\frac{\partial(\alpha_f h)^2}{\partial x}. \tag{13}$$

Here, $\alpha_f h = H$ is realized as the effective fluid table (or the amount of fluid at a given position) in the porous media. So, with Eqs. (12) and (13), Eq. (11) can be written as:

$$\frac{\partial H}{\partial t} - D\frac{\partial^2 H^\alpha}{\partial x^2} + C\frac{\partial H^\alpha}{\partial x} = 0, \tag{14}$$

where $\alpha=2$, and $D=\gamma\cos\zeta/(2C_{DG})$, and $C=\gamma\sin\zeta/(C_{DG}h)$ are called the sub-diffusion and sub-advection coefficients (or, simply diffusion and advection coefficients). Eq. (14) can be used to obtain time and spatial evolution of the fluid flow (H) through the porous media, or the porous landscape, and debris material.

As it appears, in the form, Eq. (14) is the generalized porous media equation (Boussinesq, 1903; Muskat, 1937; Leibenzon, 1930, 1945; Barenblatt, 1952, 1953; Vazquez, 2007; Evans, 2010; Vazquez, 2007, 2011). We call Eq. (14) the *sub-diffusive and sub-advective model*. As H evolves, Eq. (14) describes *a sub-diffusive and sub-advective flow* of fluid through porous and debris material. In hydrodynamic limit ($\alpha=2$) this equation is similar in form to the Lutsko and Boon (2007) model.

Special features of the new sub-diffusion and sub-advection model

In the sub-diffusive and sub-advective model (Eq. (14)), constructed above, the collective parameters D and C are explicitly expressed in terms of the density ratio, slope angle, and the generalized drag, C_{DG}, which already contains several physical parameters (Pudasaini, 2012). Importantly, in Eq. (14), the fluid is advecting due to the slope-induced gravity force. Also, the diffusion and advection coefficients are proportional to the density ratio and the slope induced gravity components $\cos\zeta$ and $\sin\zeta$, and inversely proportional to the drag coefficient.

From a physical point of view, there are several new and important aspects of the model (Eq. (14)). (i) First, we took the great advantage to have $\partial h/\partial x$ in u_f (with known power) and $\partial(\alpha_f h)/\partial t$ in mass balance over the classical porous media equation (Boussinesq, 1903; Leibenzon, 1930, 1945; Muskat, 1937; Barenblatt, 1952, 1953; Whitaker, 1986; Vazquez, 2007, 2011; Evans, 2010). Because, in our model derivation (Eq. (14)), with this, no extra closure is needed to obtain a general relationship (e.g., between u_f and the pressure gradient, $\partial h/\partial x$), which was needed in the derivation of the classical porous media equation; namely, a relationship between pressure and density. (ii) Second, in the derivation of Eq. (14), the flux exponent $\alpha=2$ characteristically corresponds to the fluid flow through porous and debris material as derived from the two-phase mass flow model (Pudasaini, 2012). In general, in the classical porous media equation (which corresponds to Eq. (14) with $C=0$) usually α is not known, but can be used as a fit parameter. However, in our derivation of Eq. (14), $\alpha=2$ emerges explicitly and systematically from the underlying mixture model. It is important to note that, for sub-diffusion only flows ($C=0$) the model similar to Eq. (14) can also be obtained by using Dupuit's pioneering assumption: mainly the supposition of a very small slope of the preatic surface (Dupuit, 1863; Bear, 1972). This results in a widely used simple sub-diffusion model in which $\alpha=0$ appears by construction. Nevertheless, with respect to the relevant variable ($H=\alpha_f h$) we do not necessarily require Dupuit's postulate to derive Eq. (14). Classically, the Darcy law together with Dupuit's hypothesis leads to a sub-diffusion equation (Dupuit, 1863; Bear, 1972), but our model is general and covers both the sub-diffusion and sub-advection phenomena that can be applied to both flows in horizontal and inclined configurations, and is derived for more general settings from a completely new perspective of a mixture model. As mentioned earlier, the sub-diffusion and sub-advection coefficients (C and D) contain several physical parameters (e.g., via generalized drag, C_{DG}, which models fluid flows through both the densely and loosely packed porous media) of the solid matrix and the fluid — significantly more than in a classical Darcy flow that contains only the hydraulic conductivity. A particular case of our new equation is the Dupuit-type classical model. An explicit form of Eq. (14) with $\alpha=0$ that appears both in sub-diffusion and sub-advection does not exist in literature. So, we have fundamentally new contributions in several aspects. (iii) Third, the advection in the new model is associated with the slope. For horizontal landscape and debris material profile, the fluid can only diffuse, but does not advect. Here, $C=0$ is equivalent to $\zeta=0$, which means the debris material is not inclined and there is no gravity to pull the fluid down. So, there is no advection. As slope increases (relatively) diffusion decreases and advection increases. So, for smaller slopes, diffusion dominates, whereas for larger slopes advection dominates which is intuitively clear. (iv) Fourth, larger values of the drag, C_{DG}, intrinsically imply more dense material with effectively low permeability (Pudasaini, 2012) resulting in hindered diffusion and advection processes. The structures of sub-diffusion and sub-advection coefficients D and C show that with increasingly dense material, the diffusion and advection processes become slow, which is in line with the physics of fluid flow in porous media. These four aspects can be observable phenomena in nature. These important (structural) observations present new insights in understanding the basic nature of the flow of fluid in the porous and debris material, natural landscape and lateral embankments.

It is important to note that, as mentioned, the structure of the model equation in the form (Eq. (14)) is not fully new. However, the way it is derived, its generality, the appearance and explicit definitions of physical parameters C,D and α are new. Hence, from the physical point of view, the deterministic model (Eq. (14)), which describes the sub-diffusion and sub-advection of the fluid in general porous debris material and inclined landscape, is new.

4. Exact analytical and numerical solutions for sub-diffusion model

For $\alpha=1$, exact solutions can be constructed for Eq. (14). For either $C=0$ (diffusion only), or $D=0$ (advection only) general exact solutions can be constructed for Eq. (14). However, if $\alpha\neq 1$, it is a great mathematical challenge (even for $\alpha=2$, our particular interest) to construct exact solutions, when $D\neq 0, C\neq 0$. In what follows, we construct some exact or, analytical solutions for Eq. (14), or its reduced variants.

4.1. Classical diffusion and advection

Assume that $\alpha = 1$, $C \neq 0$ and $D \neq 0$. This corresponds to the classical linear (with respect to diffusion and advection fluxes) diffusion–advection equation (Socolofsky and Jirka, 2005; Cushman-Roisin and Becker, 2011). Exact analytical solutions exist for this. However, the question remains: if this solution is physically meaningful for the fluid flows through porous and debris materials. The analysis below addresses this issue.

4.2. Generalized diffusion, porous media equation

Next, assume that $\alpha = 2$, $C = 0$, and $D \neq 0$. Then Eq. (14) reduces to a porous media equation:

$$\frac{\partial H}{\partial t} - D\frac{\partial^2 H^\alpha}{\partial x^2} = 0, \qquad (15)$$

which is similar to Richards' equation for water flow in unsaturated soil (Richards, 1931), where $H(=\theta)$ is the moisture content (percentage of pore space at location x filled with fluid at time t), or the porous media flow model developed on the basis of Dupuit's hypothesis and Darcy's law (Dupuit, 1863; Bear, 1972).

The exact solution to Eq. (15) exists and can be written in the form of the q-Gaussian (Barenblatt, 1952, 1953; Boon and Yip, 1991; Boon and Lutsko, 2007; Lutsko and Boon, 2007; Vazquez, 2007, 2011):

$$H(t,x) = t^{-\gamma/2}\left[\mathcal{B}^{\frac{1}{1-q}}\left(1-\frac{1}{2m_0\mathcal{B}D}\frac{(1-q)}{(3-q)}\frac{x^2}{t^\gamma}\right)^{\frac{1}{1-q}}\Theta\left(1-\frac{1}{2m_0\mathcal{B}D}\frac{(1-q)}{(3-q)}\frac{x^2}{t^\gamma}\right)\right], \qquad (16)$$

where Θ is the step function and m_0 is a constant. In Eq. (16), $\mathcal{B}$ is the normalization factor given by

$$\mathcal{B} = \left(\frac{\eta(\eta+2)}{8m_0 D}\right)^{\frac{\eta}{\eta+2}} B\left(\frac{1}{\eta},\frac{1}{2}\right)^{-\frac{2\eta}{\eta+2}}, \qquad (17)$$

where $\eta = 1-q$, $\gamma = 2/(2+\eta)$, B is the beta function, and $q \to 0$ corresponds to $\alpha \to 2$. In the limit, $q \to 1$, Eq. (16) with $C = 0$ approaches the classical Gaussian solution. In contrast to the classical Gaussian solution, Eq. (16) has some special properties: (i) These solutions have compact supports. (ii) The front travels with finite speed, unlike other parabolic solutions (Smyth and Hill, 1988; Vazquez, 2007, 2011). The interface between the solution and the outer domain ($\Gamma = \partial(\overline{suppH})$) is a free boundary, and that near the interface the solutions are only of class C^α (α times continuously differentiable), for some α (Daskalopoulos, 2009).

We mention that the solution in the form Eq. (16) is not new. Nevertheless, as explained above, our understanding of the flow of fluid in porous landscape and debris material as sub-diffusive process as described by Eq. (16) is fundamentally new. This is so, because, our new model contains new structure and well constrained diffusion and advection coefficients, and flux exponent (D, C, α) in terms of known physical parameters associated with the solid and fluid phases of the debris material. This is not the case for the classical porous media equation.

Fig. 2 plots the time evolution of diffusion with $D = 0.1329$, $\alpha = 2$, and $m_0 = 2$. Exact solutions, as given by Eq. (16), are in solid lines and the high-resolution TVD-NOC (Tai et al., 2002; Pudasaini and Hutter, 2007; Pudasaini and Miller, 2012; Pudasaini and Krautblatter, 2014) numerical solutions for Eq. (15) are plotted with open circles. Numerical solutions fit exactly with analytical solutions. This validation allows for the legitimate numerical solutions of the full general sub-advection and sub-diffusion process (Eq. (14)). As Fig. 2 shows, the solution to sub-diffusive fluid flow in porous media is fundamentally different from the diffusive fluid flow or diffusion of heat, tracer particles and pollutant in fluid. In the latter case, the diffusion (flux) is linear (i.e., $\alpha = 1$), and the solution is represented by the classical Gaussian distribution. In the sub-diffusive process (here, $\alpha = 2$), the fluid diffuses slowly in time, and thus to maintain the mass balance, the flow (substance) is less spread (in the lateral direction).

We mention that, since this paper focuses only on model derivation and construction of exact, analytical, and numerical simulations but not on the comparison between the model solution and the data, the explanation of the chosen parameter values (for D, C, m_0, etc.) is not essential. Suitably chosen parameter values may suffice for our purpose.

The sub-diffusive solution ($q = 0$), approaching the classical Gaussian solution by the sub-diffusive solution ($q \to 1$), and the classical Gaussian solution are plotted together in Fig. 3. The chosen parameter values are $D = 0.3542$, $t = 500$, and the constant of integration (Boon and Lutsko, 2007) being unity. The initial profile ($t = 0$) is a point source of unit mass. There is a large difference between the quadratic sub-diffusive solution and solution with classical linear diffusion model. But, in the limit, as $q \to 1$, sub-diffusive solution exactly recovers the classical Gaussian solution. The solution with quadratic flux is more ellipsoidal rather than Mexican-hat type, corresponding to the solution of the classical diffusion equation with linear flux. Also note that, with the sub-diffusive model, the diffusion and dispersion is much slow and much less spread as compared to the same with the linear flux (the classical diffusion model). This can be explained and visualized as follows: The time scale for the quadratic flux is fundamentally different from the time scale in the classical (linear flux) diffusion equation. This ultimately leads to two completely different solutions as presented in Fig. 3. The new sub-diffusive solution reveals that the flow of fluid through porous and debris material should be modeled with the non-linear (quadratic) diffusion flux rather than the classical linear diffusion model. With regard to the underlying physics, such a special phenomenon for generalized porous medium flow is revealed here for the first time for the flow of fluid in the debris bulk, or porous landscape. The same model may be applied for the diffusion of fluid through skin and bone tissues in bio-medical applications with suitable values of D (and α).

4.3. Burger's equation

Further, assume that $\alpha = 2$, $C \neq 0$, and $D = 0$. Then Eq. (14) becomes Burger's equation (Burgers, 1948; Hopf, 1950; Cole, 1951; Evans, 2010; Pudasaini, 2011) with wave speed $2CH$. For this, exact solutions can be constructed in different forms and situations, including the rarefaction $H = \left(\frac{1}{2C}\right)\frac{x}{t}$.

4.4. Sub-diffusion and sub-advection with quadratic fluxes

High resolution numerical solutions for sub-diffusion model has been computed and validated with exact solutions in Fig. 2. This paves the way for the numerical solution of full sub-diffusion and sub-advection model (Eq. (14) with $\alpha = 2$, which is the most interesting aspect, as it emerges here from the derivation of the model equation). The results can be compared with the similar Monte Carlo simulation in Boon and Lutsko (2007). This, then, legitimately explains the full and general behavior of sub-diffusive and sub-advective fluid flow in porous and debris material as modeled by Eq. (14). This will present a novel picture.

5. Numerical solutions of sub-diffusion and sub-advection model

As mentioned, the drag can be linear ($j = 1$) or quadratic ($j = 2$), then from Eq. (10)

$$u_f = \left[\frac{\gamma\cos\zeta}{C_{DG}}\alpha_f\left(\tan\zeta - \frac{\partial h}{\partial x}\right)\right]^{1/j}. \qquad (18)$$

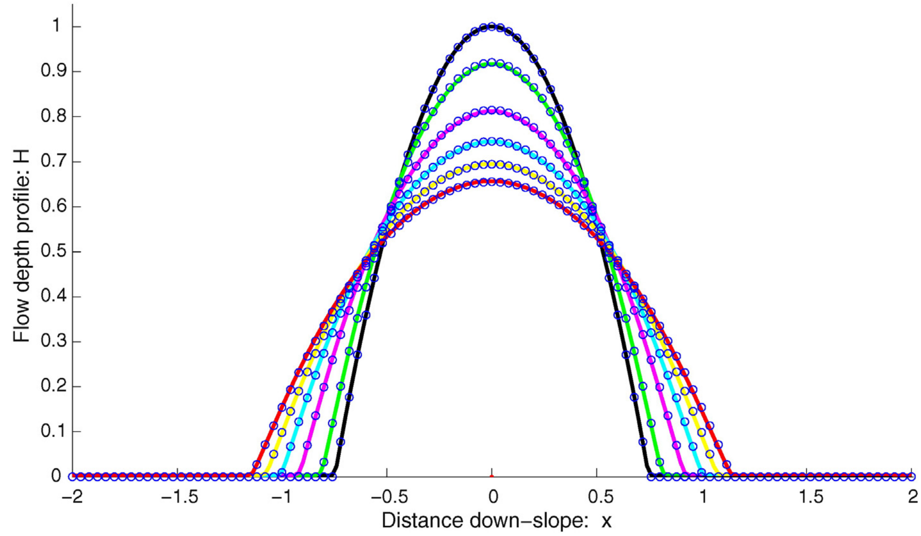

Fig. 2. Time evolution of the sub-diffusive fluid flow in porous and debris material ($\alpha = 2$). The exact solutions, as given by Eq. (16) (solid lines) are compared with the numerical solutions of Eq. (15) (open circles). The solution is presented for time slices $t_0 + [0.1, 0.3, 0.5, 0.7, 0.9]$, dark, green magenta, cyan, yellow and red, respectively, where $t_0 = 0.352$ is the reference time with $H_0 = H(t_0, x)$, the dark curve. Numerical solutions match perfectly with exact solutions. (For interpretation of the references to colors in this figure legend, the reader is referred to the web version of this article.)

Then, the mass balance for fluid (Eq. (8)) becomes

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left(\left[CH^{j+1} - \frac{D}{j+1}\frac{\partial H^{j+1}}{\partial x}\right]^{1/j}\right) = 0, \tag{19}$$

where, in the present form, $D = \gamma \cos\zeta / C_{DG}$. This can be written in more convenient form as follows:

$$\frac{\partial H}{\partial t} + \frac{1}{j}\left[CH^{j+1} - \frac{D}{j+1}\frac{\partial H^{j+1}}{\partial x}\right]^{\frac{1}{j}-1}\left[C\frac{\partial H^{j+1}}{\partial x} - \frac{D}{j+1}\frac{\partial^2 H^{j+1}}{\partial x^2}\right] = 0. \tag{20}$$

Note that, for $j = 1$, the first square bracket becomes unity, and Eq. (20) reduces to Eq. (14) with $\alpha = 2$.

Even for $C = 0$, Eq. (20) may not be solvable analytically. However, this can be solved numerically. Eq. (20) can be properly simplified to reduce derivatives of higher order terms, e.g., $\partial H^{j+1}/\partial x = (j+1)H^j \partial H/\partial x$. The second square bracket in Eq. (20) can also be written in a better form as

$$\left[C\frac{\partial H^{j+1}}{\partial x} - \frac{D}{j+1}\frac{\partial^2 H^{j+1}}{\partial x^2}\right] = \frac{\partial}{\partial x}\left[CH^{j+1} - \frac{D}{j+1}(j+1)H^j\frac{\partial H}{\partial x}\right]. \tag{21}$$

Numerical solutions for Eq. (20), with $C = 2.5, D = 0.1329$, and $j = 1$ (linear drag), and $j = 2$ (quadratic drag) are shown in Figs. 4 and 6, respectively. Fig. 5 represents the classical advection and diffusion solution.

Fig. 4 reveals a very special sub-advective and sub-diffusive flow behavior: the flow becomes more sharp (bore-type) in the front and it also elongates in the downslope direction. The front advects continuously. However, the tail remains effectively unmoved.

This is a typical behavior of sub-advective flow (Pudasaini, 2011). Since the flow is sub-diffusive, the flow does not spread laterally (here along x) which would be the case in the classical diffusion and advection processes. This solution is completely different from the solution for the classical diffusion–advection equation, where the entire fluid pocket would advect in the downslope direction, which at the same time also diffuses with spreading Gaussian profile (Fig. 5). In contrast to the classical advection–diffusion of fluid in open environment, or transport of

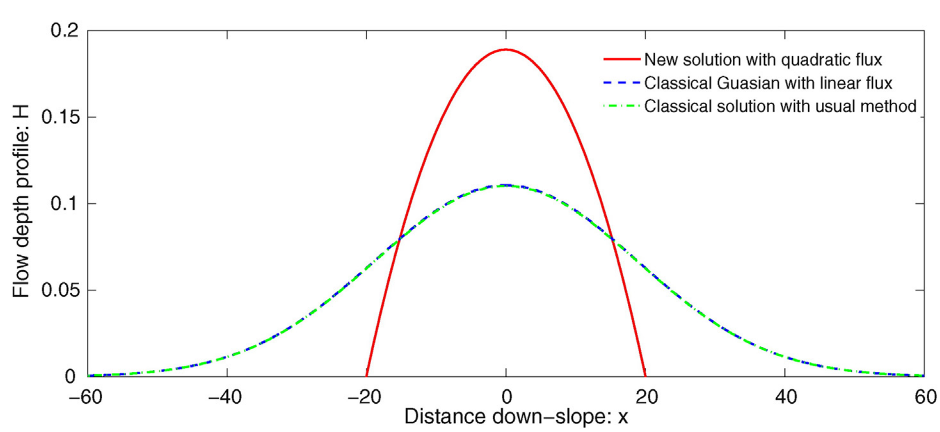

Fig. 3. The sub-diffusive solution (solid red line, $q = 0$, i.e., quadratic flux, $\alpha = 2$), approximation of the classical Gaussian solution by the sub-diffusive solution (dashed blue line, $q \to 1$) and the classical Gaussian solution (dashed green line) are plotted together. There is a large difference between the sub-diffusive solution and solution with classical (linear flux) diffusion model. (For interpretation of the references to colors in this figure legend, the reader is referred to the web version of this article.)

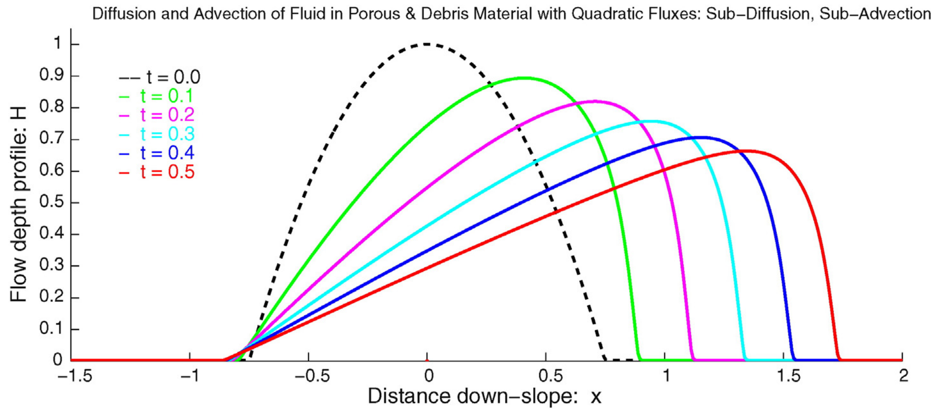

Fig. 4. Evolution of the sub-diffusion and sub-advection of a fluid through a debris and porous media. The initial condition is indicated by the dashed curve at $t = 0$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

tracer particles and other substances by transporting fluid, where the tail of the initial substance distribution also advects (with the speed of the transporting background fluid) in the downslope direction, here for the viscous fluid flow through porous media, the tail, although less in intensity, always remains in its original position (Fig. 4). This is so, because, some viscous fluid must stick at its original position. This is the most likely scenario for fluid flow in porous media. This is due to the fact that here (in Fig. 4) the solid porous skeleton is effectively stationary (unlike the moving background fluid that transports tracer particles in the usual advective–diffusive flow) and, here (in Fig. 4), the advection of the fluid is due to the down-slope gravity force on the fluid.

The solution similar to that in Fig. 4 has also been presented by Lutsko and Boon (2007) for their sub-diffusion, sub-advection model that emerged from generalized form of the classical Fokker–Planck equation (Risken, 1989) with the effect of drift. They solved their model by applying the Monte Carlo numerical simulation technique. The similarity in the numerical solution of sub-diffusive and sub-advective, in our result (Fig. 4), and that in Lutsko and Boon (2007), is that the solution is more and more asymmetric as time progresses. However, the underlying physics are different.

As shown in Fig. 5 for usual advection–diffusion, the flow immediately assumes classical Gaussian type shape. Observe that, in contrast to sub-diffusive and sub-advective model (Fig. 4), in Fig. 5, both the flow front and the tail are moving in the down-slope direction, and that the profile is diffusing in both the down-slope and up-slope directions. This is not a likely scenario in the flow of a viscous fluid through a porous media. The flow profile evolution in Fig. 5 is completely different from that in Fig. 4. So, the physical mechanisms of the two flow situations are different. This indicates that the fluid flows through the porous media and debris material should be described by sub-diffusive and sub-advective model rather than classical diffusion–advection model.

Fig. 6 displays the effect of the quadratic drag in the sub-diffusive and sub-advective fluid flow in porous and debris material. Quadratic drag increases the exponents in the sub-advective and sub-diffusive fluxes, and effectively reduces the wave speed (of fluid flow). This resulted in the slow motion and deformation. As compared with the linear drag (Fig. 4), with the quadratic drag (Fig. 6), the front is much steeper, flow depth is larger, the fluid body is relatively amplified in the tail, the solution there is also strongly curved, and the flow is less stretched. The numerical solutions show that the generalized drag plays important role in determining the form and propagation speed of the diffusion–advection waves. Such a typical behavior could not be described by the linear drag and the models that are derived with Dupuit's assumption, and linear Darcy law resulting in a simplified and reduced sub-diffusive flow.

Numerical solutions reveal that results for the sub-diffusion and sub-advection are largely independent of their initial profiles as shown in Fig. 7, where the considered initial profiles were triangular (dashed line), Gaussian (solid line), and rectangular (dashed–dotted line). However, as evident from the initial profiles, the solutions with the Gaussian profile mainly lies in between the rectangular and triangular (in the front) and triangular and rectangular (in the back) solutions. So, as the initial fluid profiles do not substantially influence the time evolution of the fluid profiles through the porous and debris material, the accurate knowledge of the initial fluid distribution may not be necessary for the

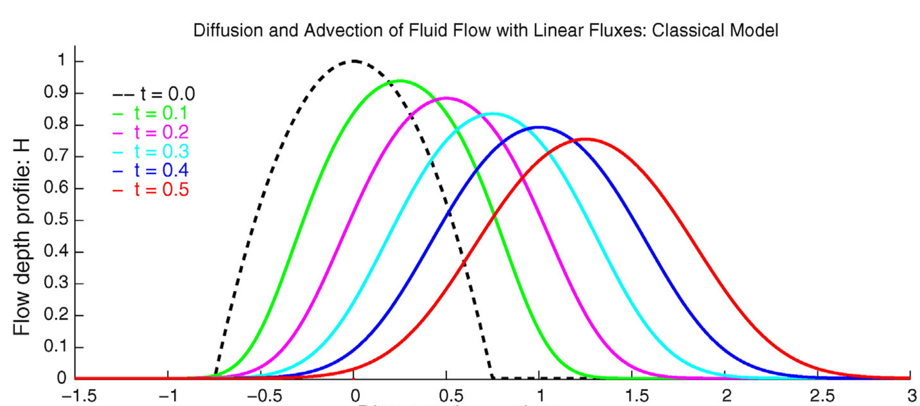

Fig. 5. Same as in Fig. 4, but now with linear diffusion–advection fluxes, the classical model. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

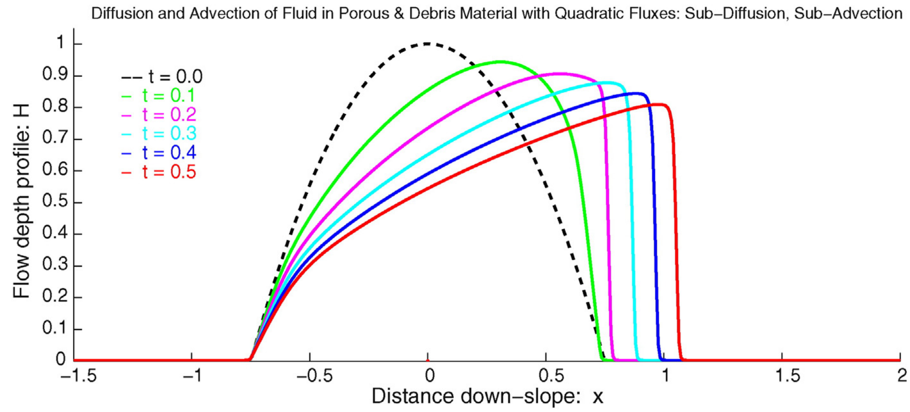

Fig. 6. Same as in Fig. 4, but now with quadratic drag, $j = 2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

long-time evolution of the sub-diffusive and sub-advective flows. This is an important and novel information in understanding the intrinsically complex behavior of diffusive and advective fluid flow in porous and debris materials.

6. Analytical solution to sub-diffusion and sub-advection flow

It is a great challenge to construct exact analytical solutions, even to the seemingly simple sub-diffusion and sub-advection Eq. (14) with $\alpha = 2$. We outline some possible ways to construct exact solutions as they play important role to gain insights into the full system and to validate numerical solution methods and results.

6.1. Transformation to classical diffusion and advection equation

With the successive substitutions $\mathrm{H}^2 = \mathcal{H}$, and $2\sqrt{\mathcal{H}}\partial t = \partial\tau$, (Eq. (14)) (with $\alpha = 2$) formally transforms, in the form, to the usual diffusion–advection equation for $\mathcal{H}$ in τ and x

$$\frac{\partial\mathcal{H}}{\partial\tau} - D\frac{\partial^2\mathcal{H}}{\partial x^2} + C\frac{\partial\mathcal{H}}{\partial x} = 0. \tag{22}$$

Exact solution for Eq. (22) exists and takes the classical Gaussian-form (for simplicity, with reference time $\tau_0 = 0$, and point source location at $x_0 = 0$),

$$\mathcal{H}(\tau, x) = \frac{\mathcal{M}}{\sqrt{4\pi D\tau}}\exp\left[-\frac{(x - C\tau)^2}{4D\tau}\right], \tag{23}$$

where $\mathcal{M}$ is the mass that is instantaneously released from a point source. Since $\mathcal{H}$ is known in terms of τ, we may use $\partial t = (1/2\sqrt{\mathcal{H}})\partial\tau$ to find a functional relationship for τ in terms of t. However, although desirable, it may be very difficult, if not impossible, to explicitly and fully convert the analytical solution Eq. (23) in terms of the original variables t and H. Here, we construct some simple analytical solutions. For simplicity, we set $\xi = x - Ct$, which is a new spatial variable. With this, $\partial t = (1/2\sqrt{\mathcal{H}})\partial\tau$ reduces to:

$$\partial t = \frac{1}{K_2}\left(\tau^{1/4}\exp(K_1/\tau)\right)\partial\tau, \tag{24}$$

where $K_2 = 2\mathcal{M}^{1/2}/(4\pi D)^{1/4}$ and $K_1 = \xi^2/8D$. Eq. (24) possesses an exact solution:

$$t = \frac{4}{5K_2}\left[\exp(K_1/\tau)\left\{4K_1\tau^{1/4} + \tau^{5/4}\right\} + \frac{4K_1^2\Gamma\left(\frac{3}{4}, \frac{-K_1}{\tau}\right)}{\tau^{3/4}\left(-\frac{K_1}{\tau}\right)^{3/4}}\right], \tag{25}$$

where Γ is the gamma function. However, it is still difficult to obtain an explicit expression for τ in terms of t in Eq. (25), which may demand a very complicated inversion technique if it is possible at all to do so.

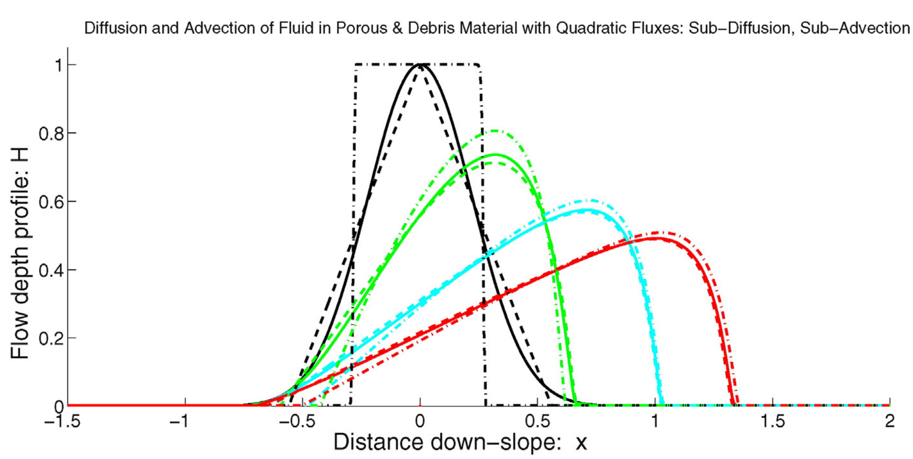

Fig. 7. Sub-diffusive and sub-advective fluid flow through porous media with three different initial fluid profiles: smooth Gaussian (solid line), triangle (dashed line) and rectangle (dashed-dotted line). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Nevertheless, due to the presence of τ in the denominator of exponential and gamma functions, Eq. (25) can be largely simplified. For small values of τ, the factor $\Gamma(3/4, -K_1/\tau)$ becomes almost zero, and the remaining expression can be simplified when τ is sufficiently away from origin. For sufficiently large values of τ, $\Gamma(3/4, -K_1/\tau)$ is bounded approximately by 1.225 and $\exp(K_1/\tau)$ by unity. But, $4K_1\tau^{1/4}+\tau^{5/4}$ increases monotonically and dominates other terms. Therefore, for such values of τ, Eq. (25) can be approximated by

$$t \approx \frac{4}{5K_2}\left[4K_1\tau^{1/4}+\tau^{5/4}\right]. \tag{26}$$

The lower and higher order expressions in Eq. (26) can be utilized separately. As we will see later, such approximation may capture basic features that are of interest to us.

First, we deal only with the lower order term in τ in Eq. (26). For this, we obtain a simple explicit solution that may be useful

$$\tau = \left[\frac{5}{16}\frac{K_2}{K_1}t\right]^4. \tag{27}$$

The advantage with this solution is that it includes both K_1 and K_2. Solution (27) can be written in the original variable (coordinate) and in terms of D and C as:

$$\tau = \frac{5^4\mathcal{M}^2D^3}{4\pi}\left[\frac{t}{(x-Ct)^2}\right]^4. \tag{28}$$

A simple analytical solution for Eq. (14) is then given explicitly by Eqs. (23) and (28), and $H^2=\mathcal{H}$.

Fig. 8 shows simple analytical solutions for the time evolution of the sub-diffusive and sub-advective fluid flow in porous media with $\mathcal{M}=2.9, C=7.64\times10^{-06}$, and $D=0.2167$. Note that such small value of C is needed mainly to make the solution smooth and to be attached at the tail for all the times. Otherwise, and for small times, the value of C can be taken four or more higher order of magnitude. The initial geometry is a point source just to the right of $x_0=0$.

Comparing Figs. 4 and 5, we observe that Fig. 8 presents reasonable results. Clearly, there are large differences between the classical diffusion–advection solution and the present solution for the sub-diffusion and sub-advection process for the fluid flow in the porous and debris material. Importantly, the solutions in Fig. 8 are non-symmetrical which is one of the main characteristics of the sub-diffusive and sub-advective flows.

As indicated by Eq. (25) most probably there exists no simple expression to fully express τ in terms of t. Nevertheless, it is important to note that solution (27) provides some potentially useful expression revealing that in lower order approximation, τ varies with the fourth power of t, i.e., $\tau \sim t^4$. The usefulness of this approximation is revealed by Fig. 8 that plots associated solution, which qualitatively, is in line with the high-resolution numerical solution of full sub-diffusion and sub-advection model as presented in Fig. 4.

However, further sophisticated and perhaps improved solutions can be constructed by considering both terms in Eq. (26). Then, with the substitutions, $\tilde{t}=-5K_2t/4$, $a=4K_1$, and $\tau^{1/4}=T$, Eq. (26) takes the form of a reduced quintic polynomial, in the Bring–Jerrard normal form

$$T^5+aT+\tilde{t}=0. \tag{29}$$

The root of Eq. (29) can be written in the Bring radical (ultraradical), that is expressed in terms of a hypergeometric function, ${}_4F_3$ as

$$T=\left[-\frac{a}{5}\right]^{1/4}B_R\left(-\frac{1}{4}\left[\frac{5^5}{-a^5}\right]^{1/4}\tilde{t}\right), \tag{30}$$

where B_R is the Bring radical (Bring, 1864), given by

$$B_R(\lambda)=-\lambda\,{}_4F_3\left(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5};\frac{1}{2},\frac{3}{4},\frac{5}{4};-\left[\frac{5\lambda}{4}\right]^4\right). \tag{31}$$

Time evolutions of these more advanced analytical solutions are presented in Fig. 9 with the parameter values $\mathcal{M}=35.0$, $D=2.166$, and $C=1.1461$.

As in Fig. 8, the solutions in Fig. 9 are non-symmetrical. The bore-type flow fronts are advecting in the down-slope direction, however, the rear positions remain effectively very close to its reference position. As mentioned before, these are the main characteristics of sub-diffusive and sub-advective flows. This reveals intuitively clear and the general behavior of a sub-diffusive and sub-advective fluid flow in porous media and debris material.

Although there are clear differences, comparison between Figs. 8 and 9 indicates that the rear may be better modeled by Fig. 8 whereas the front may be better described by Fig. 9. Whether the higher or the lower order solution is more appropriate should be scrutinized which is out of scope here. In general, higher order solutions are assumed to be better, but in particular situations, due to some associated fluctuations, lower order solutions may provide simple and more realistic descriptions. However, for the more complete and accurate results we

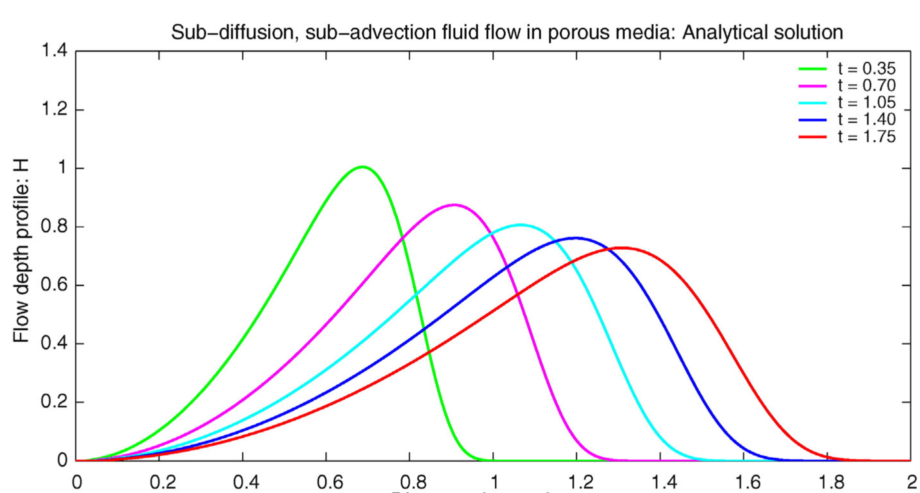

Fig. 8. Sub-diffusive and sub-advective fluid flow through porous media: simple analytical solutions. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

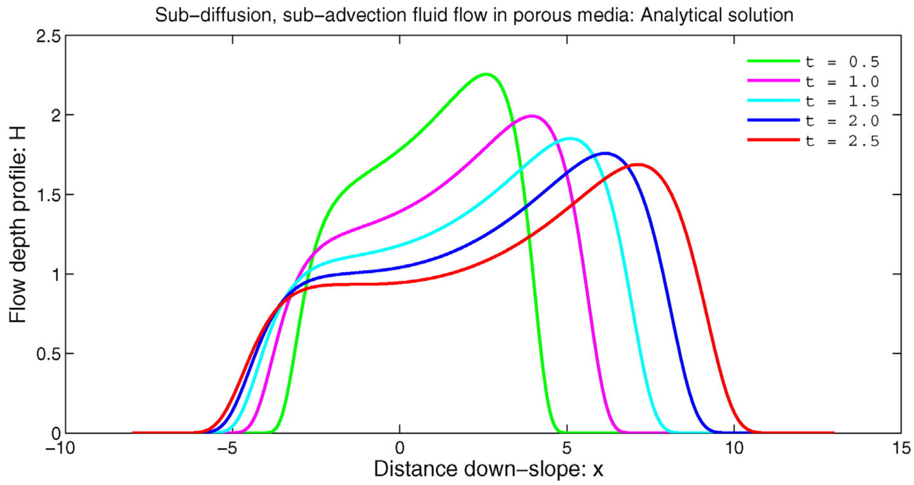

Fig. 9. Sub-diffusive and sub-advective fluid flow through porous media: advanced analytical solutions. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

must use Eq. (23) together with Eq. (25) with the full relationship between t and τ. Note that, the solution techniques presented here may, in principle, be extended for any α in Eq. (14).

6.2. Separation of variables

Derivations from Eqs. (23) to (28), and the complicated nature of Eq. (25) also suggest that it is worth seeking some other alternative methods for constructing the full exact/analytical solutions. Another important method to analytically solve the diffusion–advection equation is by the method of separation of variables. Here, we apply this method for sub-diffusion and sub-advection problem. Assume the separation:

$$H(t,x) = \frac{X(x)}{T(t)}. \tag{32}$$

Then, Eq. (14) (with $\alpha = 2$) reduces to

$$\frac{1}{D}\frac{\partial T}{\partial t} = \frac{1}{X}\left[\mathcal{C}\frac{\partial X^2}{\partial x} - \frac{\partial^2 X^2}{\partial x^2}\right], \tag{33}$$

where, $\mathcal{C} = C/D$. In Eq. (33), the left hand side is only a function of t and the right hand side is an expression only in x. Thus, the variables are separated. Eq. (33) implies that there must exist a constant K such that,

$$\frac{1}{D}\frac{\partial T}{\partial t} = -K = \frac{1}{X}\left[\mathcal{C}\frac{\partial X^2}{\partial x} - \frac{\partial^2 X^2}{\partial x^2}\right]. \tag{34}$$

The constant K should be determined from the solution for T and X, and other conditions associated with the model equation. This equation now represents two ordinary differential equations, one for T, and the other for X. The left and middle expressions in Eq. (34) imply a solution for T:

$$T = K_1 - DKt, \tag{35}$$

where, K_1 is a constant of integration. The middle and right expressions in Eq. (34) result in a second order ordinary differential equation for X in x.

$$\frac{d^2X^2}{dx^2} - \mathcal{C}\frac{dX^2}{dx} = KX. \tag{36}$$

The great complexity in solving this equation analytically is due to the fact that the right hand side is only linear in X, whereas the unknown variable (X) is in quadratic form in the left within the differential operators.

With the substitution, $X^2 = Y$, Eq. (36) becomes,

$$Y''_{xx} - \mathcal{C}Y'_x = K\sqrt{Y}, \tag{37}$$

which is the Lienard equation. This can be reduced to the Abel equation in canonical form (Lienard, 1928; Polyanin and Zaitsev, 2003; Khattri, 2014).

Full exact solution could not be constructed for Eq. (37). We solved Eq. (37) numerically. Semi-analytical and numerical solutions for the separation of variable method are then obtained from Eqs. (32), (35), and (37). The separation of variable solution is presented in Fig. 10 for time $t = 0.5$, and parameter values $D = 0.1045$, $\mathcal{C} = 2.05$, $K = -0.103$, and $K_1 = 0.07$. It is important to note that this result is very close to the high-resolution numerical solution presented in Fig. 4. Thus, the separation of variable method also provides a reasonable, potentially useful, and physically meaningful solution to the full sub-advection and sub-diffusion model.

7. Summary

Here, we considered the two-phase solid–fluid mixture mass flow model (Pudasaini, 2012). By assuming stationary solid matrix, we derived a novel dynamical model equation to describe the sub-diffusive and sub-advective flow of fluid in porous landscape and debris materials. In contrast to the classical porous media equation, we obtained this without any extra closure between the field variables, namely, the pressure and density. In general, in the classical porous media equation, usually the flux exponent α is not known, but can be used as a fit parameter. However, in our new model, $\alpha = 2$ emerges explicitly and systematically from the underlying theory, similar to the transport equation based on the Dupuit hypothesis and Darcy law, but our new model, as it is developed for a general setting and is based on a mixture model, is more general than the Dupuit-type model. This important structure presents some new insights in understanding the basic nature of the flow of fluid in porous and debris material, natural landscape and lateral embankments. In hydrodynamic limit the new model reduces to the generalized porous media equation. Nevertheless, in the present model equation, the sub-diffusion and sub-advection coefficients (D and C) are explicitly expressed in terms of the density ratio, slope angle, and the generalized drag which already contains several physical parameters. Importantly, in our sub-diffusion and sub-advection model the fluid is advecting due to the slope-induced gravity force. The model equation can be solved analytically when one of the diffusion and advection coefficients vanishes. However, for $\alpha \neq 1$, it is a great

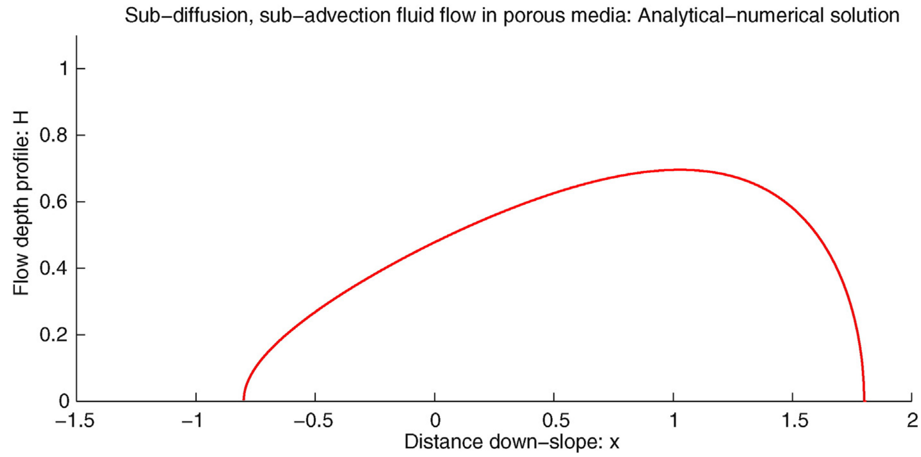

Fig. 10. Sub-diffusive and sub-advective fluid flow through porous media: semi-analytical and numerical solution as obtained by the separation of variable method. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

mathematical challenge to construct exact solutions to the sub-diffusion and sub-advection model when $D \neq 0, C \neq 0$. We have constructed some exact/analytical solutions to the full model or its reduced variants.

For vanishing advection ($C = 0$), our generalized porous media equation (sub-diffusion model) is similar, in the form, to Richard's equation for water flow in unsaturated soil. We have presented exact solutions to the generalized porous media equation with the q-Gaussian. Exact solutions for the time evolution of the sub-diffusive fluid flow in porous and debris material ($\alpha = 2$) have been compared with the high-resolution numerical solutions. Numerical solutions match perfectly with exact solutions. This validation allows for the legitimate numerical solutions of the full general sub-advection and sub-diffusion process. Our results show that the solution to sub-diffusive fluid flow in porous media is fundamentally different than the diffusive fluid flow or diffusion of heat, tracer particles and pollutant in fluid. There is a large difference between the (quadratic) sub-diffusive solution and solution with classical (linear) diffusion model. However, in the limit, sub-diffusive solution exactly recovers the classical Gaussian solution. The solution with quadratic flux is more ellipsoidal rather than Mexican-hat type, the solution of the classical diffusion equation. With the sub-diffusive model, the diffusion is much slow and much less spread as compared to the same with the linear flux. Such a special phenomenon for generalized porous medium flow is revealed here for the first time for the flow of fluid in the debris bulk and the porous landscape. The same model may potentially be applied for the diffusion of fluid through skin and bone tissues in bio-medical applications.

Next, the full sub-diffusion and sub-advection equations are solved numerically. The sub-diffusion and sub-advection solution is completely different from the solution for the classical diffusion–advection equation, where the entire fluid pocket would advect in the downslope direction, which at the same time also diffuses with spreading Gaussian profile. In contrast to the classical advection–diffusion of fluid in open environment, or transport of tracer particles and other substances by transporting fluid, where the tail of the initial substance distribution also advects in the downslope direction, here for the viscous fluid flow through porous media, the tail always remains in its original position. This is the most likely scenario for fluid flow in porous media. This is due to the fact that here the solid porous skeleton is effectively stationary (unlike the moving background fluid that transports tracer particles in usual advective-diffusive flow) and, here, the advection of the fluid is due to the down-slope gravity force on the fluid. So, physically the transport mechanisms in the two flow scenarios are different. This indicates that fluid flows through porous landscape and debris materials should be described by sub-diffusive and sub-advective model rather than the classical diffusion–advection model. Simulation results show that the accurate knowledge of initial profile of the fluid in porous media is not necessary for the long-time evolution of sub-diffusive and sub-advective flows. This novel information is important in understanding the complex behavior of fluid flow in porous and debris materials. Simulations performed with quadratic drag resulted in the slow motion and deformation. As compared with the linear drag, with the quadratic drag, the front developed into a perfect bore, and the flow is less stretched.

Although the form and structure of the model equation derived here is not fully new, the way it is derived, the appearance and explicit definitions of the physical parameters and their general forms and wide range of applicability, including the sub-diffusion and sub-advection coefficients and the flux exponent are new. Hence, from the physical point of view, our new deterministic model, which describes the sub-diffusion and sub-advection of the fluid in general porous debris material and inclined landscape, is new.

For reduced situations, simple and advanced analytical solutions, presented for the sub-diffusion and sub-advection model, reveal some potentially observable phenomena. There are large differences between the exact solutions for the classical diffusion–advection model and the new exact solutions for the sub-diffusion and sub-advection process. Although substantial differences are observed between the numerical solutions for the full model and the analytical results for the sub-diffusion and sub-advection model, the new analytical results potentially capture some basic features of fluid flows in porous landscape and debris material. We have also outlined some procedures for constructing the full analytical solutions by using the method of separation of variables. Semi-analytical-numerical solution has been presented for the model that emerged from the separation of variable method. Each of the analytical and computational results presented here is subject to scrutiny with experimental and/or field data, which, however, is not within the scope of the present work.

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