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Dispersive landslide

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ABSTRACT

Considering the non-hydrostatic mass flow model (Pudasaini, 2022 [1]), here, I derive a novel dispersive wave equation for landslide. The new dispersive wave for landslide recovers the classical dispersive water waves as a special case. I show that the frequency dispersion relation for landslide is inherently different than the classical frequency dispersion for water waves. The wave frequency with dispersion increases non-linearly as a function of the wave number. For dispersive landslide, the wave frequency without dispersion appears to heavily overestimate the dispersive wave frequency for higher wave number. Due to the dispersion term emerging from the non-hydrostatic contribution for landslide, the phase velocity becomes a function of the wave number. This gives rise to the group velocity that is significantly different from the phase velocity, characterizing the dispersive mass flow. The dispersive phase velocity and group velocity decrease non-linearly with the wave number. Yet, the group velocity is substantially lower than the phase velocity. I analytically derive a dispersion number as the ratio between the phase velocity and the group velocity, which measures the deviation of the group velocity from the phase velocity, provides a dynamic scaling between them and summarizes the overall effect of dispersion in the mass flow. The dispersion number for landslide increases rapidly with the wave number, which is in contrast to the dispersion in water waves. With the definition of the effective dispersive lateral stress, I prove the existence of an anti-restoring force in landslide. I reveal the fact that due to the anti-restoring force, landslides are more dispersive than the piano strings. So, the wave dispersion in landslide is fundamentally different than the wave dispersion in the piano string. My model constitutes a foundation for the wave phenomenon in dispersive mass flows.

1. Introduction

Landslides and debris avalanches consist of a mixture of granular materials and the fluid. There have been rapid advancements in modeling such mass movements as shallow flows [2–7]. Classically, modeling geophysical flows is based on the hydrostatic, depth-averaged mass and momentum balance equations [8]. However, in rapid mass flows down inclined slopes the gravity and the vertical acceleration can have the same order of magnitude effects demanding for the non-hydrostatic model formulation [9,10].

The Boussinesq-type water wave theory is widely used in hydraulics and water wave simulations [11]. Following the work of Boussinesq [12,13], the free surface water flow simulations are generally based on non-hydrostatic depth-averaged models. Fundamental further contributions in including Boussinesq-type non-hydrostatic and dispersive effects in water waves are also due to Serre [14], Peregrine [15], Green and Naghdi [16], and Nwogu [17]. However, for shallow granular flows, Denlinger and Iverson [9] included the effect of nonzero vertical acceleration on depth-averaged momentum fluxes and stress states while modeling granular flows across irregular terrains. This was later extended by Castro-Orgaz et al. [10] resulting in the novel Boussinesq-type theory for granular flows. Yuan et al. [18] advanced further by presenting a refined and more complete non-hydrostatic shallow granular flow model.

Pudasaini [1] extended and utilized the above mentioned ideas to the multi-phase mass flow model [7] to generate a non-hydrostatic Boussinesq-type gravity wave model for multi-phase mass flows. The new non-hydrostatic multi-phase mass flow model includes enhanced gravity and dispersive effects as in the single-phase models by Denlinger and Iverson [9], Castro-Orgaz et al. [10] and Yuan et al. [18]. However, the Pudasaini [1] model further includes interfacial momentum transfers in the non-hydrostatic Boussinesq-type model formulation representing the complex multi-phase nature of mass flow.

Here, I consider the non-hydrostatic multi-phase mass flow model [1] and reduce its complexity to a geometrically two-dimensional landslide motion as a mixture of solid particles and fluid down a slope. Then, I derive a novel dispersive wave equation for the landslide motion as a complex partial differential equation. The new equation reduces to the simple classical wave equation. The landslide dispersion relation includes different physical parameters and mechanical responses. I show that the frequency dispersion relation for landslide is essentially different than the classical frequency dispersion for water waves. As the dispersion originates from the non-hydrostatic contribution for landslide, the phase velocity becomes a function of the

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wave number, resulting in the significantly different group velocity than the phase velocity revealing an impressive relation between them. The dispersive group velocity is substantially lower than the phase velocity as both decrease non-linearly with the wave number. Analytically derived dispersion number measures the departure of the group velocity from the phase velocity, and encapsulates the overall effect of dispersion in the landslide wave dynamics. In contrast to the water wave the dispersion number for landslide increases rapidly with the wave number. Existence of an anti-restoring force in landslide proves that landslides are more dispersive than the piano strings. These are new understanding for the dispersive landslide motions.

2. A dispersive wave equation for mass flow

2.1. Balance equations for mass flow

A geometrically two-dimensional motion down a slope is considered. Let t be time, (x, z) be the coordinates and (g^x, g^z) the gravity accelerations along and perpendicular to the slope, respectively. Let, h and u be the flow depth and the mean flow velocity of the landslide along the slope. Similarly, γ, α_s, μ be the density ratio between the fluid and the particles $(\gamma = \rho_f/\rho_s)$, volume fraction of the solid particles (coarse and fine solid particles), and the basal friction coefficient ($\mu = \tan\delta$, where δ is the basal friction angle of the solid particles) in the mixture material. Furthermore, K is the earth pressure coefficient, and C_{DV} is the viscous drag coefficient.

I start with the non-hydrostatic multi-phase mass flow model [1]. The model considers the vertical momentum equation, assumes the shallowness of the flow depth and the constant velocity profiles of the horizontal velocity components. It incorporates the enhanced gravities and the dispersion relations and signifies the highly non-linear, non-hydrostatic (dispersion) contributions. Reducing the sophistication, I consider a landslide motion as an effectively single-phase mixture of solid particles and fluid down a slope. This leads to a single mass and momentum balance equation describing the motion of a landslide (or a mass flow) with the non-hydrostatic contributions as [1]:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \tag{1}$$

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \left[\left\{\left((1-\gamma)K + \gamma\right)\alpha_s + \left(1-\alpha_s\right)\right\} g^z\right.\\
&\left.+ \alpha_s\left\{\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) w + C_{DV}\, wu\right\}\right]\frac{\partial h}{\partial x}\\
&+ \frac{1}{h}\frac{\partial}{\partial x}\left[\left\{\alpha_s(K-1)+1\right\}\left[\frac{h^3}{12}\left\{\left(\frac{\partial u}{\partial x}\right)^2 - \frac{\partial}{\partial x}\frac{\partial u}{\partial t} - u\frac{\partial^2 u}{\partial x^2} - 2C_{DV}\, u\frac{\partial u}{\partial x}\right\}\right.\right.\\
&\left.\left.+ \frac{h^2}{2}\left\{\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) w + C_{DV}\, wu\right\}\right]\right]\\
&= g^x - \mu\alpha_s\left[(1-\gamma)g^z + \left\{\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) w + C_{DV}\, wu\right\}\right] - C_{DV}\, u^2.
\end{aligned} \tag{2}$$

The second term on the left hand side of (2) describes the advection, while the third term (in the first square bracket) describes the extent of the local deformation that stems from the hydraulic pressure gradient of the free-surface of the landslide in which $\left(1-\alpha_s\right) g^z\partial h/\partial x$ emerges from the hydraulic pressure gradient associated with possible interstitial fluids in the landslide, and the terms associated with w are from the enhanced gravity [1]. The fourth term on the left hand side (in the second square brackets) are extra addition in the flux due to the non-hydrostatic contributions. Moreover, the third and fourth terms on the left hand side, and the other terms on the right hand side of (2) represent all the involved forces. The first and second terms on the right hand side of (2) are the gravity acceleration, effective Coulomb friction that includes lubrication $(1-\gamma)$, liquefaction $\left(\alpha_s\right)$ (because, if there is no or substantially low amount of solid, the mass is fully liquefied, e.g., lahar flows), the third and fourth terms with w emerge from enhanced gravity, and the fifth term is the viscous drag, respectively. The term with $1-\gamma$ or γ originates from the buoyancy effect. By setting $\gamma = 0$ and $\alpha_s = 1$, we obtain a dry landslide, grain flow, or an avalanche motion. However, I keep γ and α_s also to include possible fluid effects in the landslide (mixture). Note that for $K = 1$ (which may prevail for extensional flows, [8]), the third term on the left hand side associated with $\partial h/\partial x$ simplifies drastically, because $\left\{\left((1-\gamma)K+\gamma\right)\alpha_s + \left(1-\alpha_s\right)\right\}$ becomes unity. So, the isotropic assumption (i.e., $K = 1$) loses some important information about the solid content, the buoyancy effect, liquefaction and lubrication in the mixture. Furthermore, $w = -\frac{h}{2}\frac{\partial u}{\partial x}$ is the mean slope normal velocity [1,18].

2.2. Linearized mass and momentum balance equations

I linearize (1) and (2) with $h = H + \tilde{h}$, where H is the background (mean) material depth on which the amplitude $\tilde{h}$ is defined. For simplicity, the tildes are discarded from the resulting equations. Then, I obtain the linearized mass and momentum equations as:

$$\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} = 0, \tag{3}$$

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \left[\left((1-\gamma)K+\gamma\right)\alpha_s + \left(1-\alpha_s\right)\right] g^z\frac{\partial h}{\partial x} - \frac{H^2}{3}\left[\alpha_s(K-1)+1\right]\frac{\partial^2}{\partial x^2}\frac{\partial u}{\partial t}\\
&= g^x - (1-\gamma)\alpha_s\mu g^z + \frac{1}{2}\mu\alpha_s H\frac{\partial}{\partial t}\frac{\partial u}{\partial x},
\end{aligned} \tag{4}$$

where, out of $\frac{H^2}{3}\left[\alpha_s(K-1)+1\right]\frac{\partial^2}{\partial x^2}\frac{\partial u}{\partial t}$ the factors $\frac{1}{4}$ and $\frac{1}{12}$ stem from the enhanced gravity (or hydraulic pressure gradient) and dispersion, respectively. Note that, (4) extends the Peregrine [15] dispersive system [19,20] from water waves to mixture debris waves.

2.3. A novel dispersive wave equation for landslide

Now, utilizing (3), the third terms on both sides of (4) can be written in terms of h, and the resulting momentum equation yields:

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \left[\left((1-\gamma)K+\gamma\right)\alpha_s + \left(1-\alpha_s\right)\right] g^z\frac{\partial h}{\partial x} + \frac{H}{3}\left[\alpha_s(K-1)+1\right]\frac{\partial}{\partial x}\frac{\partial^2 h}{\partial t^2}\\
&= g^x - (1-\gamma)\alpha_s\mu g^z - \frac{1}{2}\mu\alpha_s\frac{\partial^2 h}{\partial t^2}.
\end{aligned} \tag{5}$$

With the help of (3), u can be removed from (5). For this, differentiate (3) with respect to t, and (5) with respect to x. Then, eliminating u from (5), I obtain a novel dispersive wave equation for landslide:

$$\frac{\partial^2 h}{\partial t^2} = \mathcal{T}\frac{\partial^2 h}{\partial x^2} + \mathcal{D}\frac{\partial^2}{\partial x^2}\frac{\partial^2 h}{\partial t^2} + \mathcal{I}\frac{\partial^2}{\partial t^2}\frac{\partial h}{\partial x}, \tag{6}$$

where $\mathcal{T}, \mathcal{D}$, and $\mathcal{I}$ are the involved physical parameters given by $\mathcal{T} = \left[\left((1-\gamma)K+\gamma\right)\alpha_s + \left(1-\alpha_s\right)\right] g^z H$, $\mathcal{D} = \left[\alpha_s(K-1)+1\right]\frac{H^2}{3}$, and $\mathcal{I} = \mu\alpha_s\frac{H}{2}$, respectively. In (6), $\mathcal{T}$ is the effective lateral stress (per unit density). For the reasons explained below, I call $\mathcal{D}$ the dispersion parameter. Here, $\mathcal{D}$ characterizes the non-hydrostatic contribution, and the term associated with $\mathcal{I}$ emerged due to the effect of enhanced gravity in the source. Note that, for a variable slope, different additional forcing terms would appear in (6), which have been neglected for now for simplicity. Eq. (6) is a complex dispersive partial differential equation for landslide. For a relatively less dense flow (i.e., substantially dilute or hyperconcentrated flows with low particle concentration) with lower friction, and/or a relatively slowly varying flow surface (i.e., $\partial h/\partial x$), the term with $\mathcal{I}$ may be ignored, e.g., consider $\mu = 0.17$ (for $\delta = 10°$), $\alpha_s = 0.2, H = 1.0$, resulting in $\mathcal{I} = 0.017$. If not, this can be revived. Eq. (6) takes the simple classical wave equation when the terms with $\mathcal{D}$ and $\mathcal{I}$ are ignored, for which $\sqrt{\mathcal{T}}$ is the wave speed for the debris motion. To explore the first order effects of the dispersive phenomena in the mixture mass flow, in what follows, for simplicity, I disregard the influence of the term associated with $\mathcal{I}$. Physically plausible values of the model parameters $\mathcal{T}$ and $\mathcal{D}$ are explained at Section 4 representing some possible scenarios.

3. Dispersion in non-hydrostatic mass flow

3.1. The dispersion relation

Assume a plain wave of the form:

$$h = h_0 \exp\left[i(kx - \omega t)\right], \tag{7}$$

where $h_0 = h(0,0)$, and ω and k are the wave frequency and the wave number ($\sim$ reciprocal of the wave length). Applying (7) in to (6) results in:

$$\omega^2 = \mathcal{T}k^2 - \mathcal{D}\omega^2 k^2, \tag{8}$$

I write (8) in the form

$$\omega^2 = \frac{\mathcal{T}k^2}{1+\mathcal{D}k^2}. \tag{9}$$

Eq. (9) is the frequency dispersion relation (linking frequency and wave number) to our model for mass flow, which is different than the classical linear frequency dispersion for Boussinesq water wave equations [21]. Note that, in (9), $\mathcal{D}$ is proportional to H^2. So, $\mathcal{D}k^2 \sim H^2k^2$, which contains the relative wave number Hk.

3.2. The phase velocity

The landslide phase velocity (speed) C_p is defined as $C_p = \omega/k$, which, from (9), takes the form

$$C_p = \frac{\omega}{k} = \pm\sqrt{\frac{\mathcal{T}}{1+\mathcal{D}k^2}}. \tag{10}$$

So, for the non-hydrostatic mass flow, the phase velocity is not a constant but is a function of the wave number. Due to the non-zero positive dispersion parameter $\mathcal{D}$, (10) gives rise to the group velocity that is different from the phase velocity. Also note that, $\mathcal{T}$ has the dimension of $\mathrm{m^2s^{-2}}$ and $\mathcal{D}k^2$ is dimensionless. Thus, C_p has the dimension of $\mathrm{ms^{-1}}$, the velocity.

3.3. The group velocity

The landslide group velocity is denoted by C_g and is defined as

$$C_g = \frac{\partial \omega}{\partial k}, \tag{11}$$

which is the measure of the rate of change of the wave frequency as a function of the wave number. From the phase velocity (10), I obtain the group velocity:

$$C_g = \frac{\partial \omega}{\partial k} = \frac{\partial}{\partial k}\left(kC_p\right) = \frac{1}{\mathcal{T}}\left(\frac{\mathcal{T}}{1+\mathcal{D}k^2}\right)^{3/2} = \left(\frac{1}{1+\mathcal{D}k^2}\right)C_p. \tag{12}$$

3.4. The dispersion number

Eq. (12) reveals a strikingly impressive relation between the phase velocity and the group velocity. There exists a function $\mathcal{D}_n^p$ of the wave number k such that it defines a mapping between the phase velocity and the group velocity given by the relation:

$$C_p = C_g\mathcal{D}_n^p, \quad \mathcal{D}_n^p(k) = 1 + \mathcal{D}k^2. \tag{13}$$

The function $\mathcal{D}_n^p$ can be written as

$$\mathcal{D}_n^p = \frac{C_p}{C_g}. \tag{14}$$

As for C_p, C_g has the dimension of velocity. So, $\mathcal{D}_n^p$, as the ratio between the phase velocity and the group velocity, is a dimensionless number. I call $\mathcal{D}_n^p$ the dispersion number that measures the deviation of the group velocity from the phase velocity. As indicated by (13), for the problem under consideration, $\mathcal{D}_n^p$ is a stretching function of the wave number and is bounded from below by unity, i.e., $\mathcal{D}_n^p \geq 1$. This means, $\mathcal{D}_n^p$ provides a dynamic scaling between C_p and C_g. It shows that $C_g \leq C_p$, which is opposite to the dispersive wave in a piano string [22,23]. Furthermore, for our problem, the phase and the group velocity have the same direction, but different speeds. Moreover, the wave becomes non-dispersive if the term associated with $\mathcal{D}$ can be ignored for which $C_g \to C_p$, consequently $\mathcal{D}_n^p = 1$, as for the ideal string, or the sound wave in a room, which are non-dispersive. However, for piano string, $\mathcal{D}_n^p$ takes the form $\mathcal{D}_n^p = \left(1+\mathcal{D}k^2\right)/\left(1+2\mathcal{D}k^2\right)$, $\mathcal{D}$ here appropriately corresponds to the physical quantity for piano. It means, for piano string the dispersion number decreases as a function of the wave number k from its maximum 1 (as $k \to 0$) to minimum $1/2$ (as k is sufficiently large). So, the wave dispersion for landslide is fundamentally different than the wave dispersion in the piano string. In other words, landslides can be more dispersive than the piano strings.

In fact, the dispersion number plays a dominant role as all the relevant quantities ω, C_p and C_g are expressed in terms of $\mathcal{D}_n^p$. Importantly, once we know $\mathcal{T}$ and $\mathcal{D}_n^p$, the wave frequency, phase and group velocities are known, because, usually, $\mathcal{T}$ is a parameter, and $\mathcal{D}_n^p$ varies as a function of the wave number.

4. Results and analyses of dispersive landslides

Here, I manifest the contribution of dispersion on the wave motion in mass flow. Unless otherwise stated, following the general values from the literature [7,8], the material parameters are chosen as follows: the earth pressure coefficient $K = 0.9$ (main downslope extensional motion), the volume fraction of solid in the mixture material $\alpha_s = 0.65$, the buoyancy (lubrication) parameter $\gamma = 1100/2900$ (ratio between the true fluid and the solid densities in the mixture), $g^z = g\cos\zeta = 6.94$ ($g = 9.81, \zeta = 45°$, the gravitational constant and the slope angle), and the mean material depth $H = 0.5$ m, respectively. Here, K represents the granular frictional behavior of the material (lower in extension, higher in compression), γ the frictional weakening due to the possible presence of the fluid, and α_s characterizes the liquefaction in the mixture material, because as $\alpha_s \to 0$ the mixture is fully liquefied [24,25]. So, α_s, K and γ together explain the behavior of the granular (debris) material in the mixture. These give the values of $\mathcal{T}$ and $\mathcal{D}$ in (10) and (12) of about 13.31 and 1.25, respectively.

4.1. The wave frequency

Fig. 1 displays the wave frequency as a function of the wave number, $\omega = kC_p$, as given by the relation (10). While the wave frequency increases linearly with the wave number without dispersion (that can be realized by setting $\mathcal{D} = 0$), the wave frequency with dispersion (including the term associated with $\mathcal{D}$ that can be realized with $\mathcal{D} \neq 0$) increases non-linearly as a function of the wave number. For small wave number both wave frequencies are similar, however, for large wave numbers, the difference is large. Furthermore, in general, the wave frequency without dispersion is much higher than the same with dispersion. When in reality the waves are dispersive, the wave frequency without dispersion appears to heavily overestimate the dispersive wave frequency for higher wave number.

4.2. The phase velocity and group velocity

The phase velocity and group velocity are technically important quantities as they provide the information of the motion of individual wave crest and energy transport of the modulated wave packet. The phase velocity without and with dispersion, and the group velocity as given by (10) and (12), respectively, are shown in Fig. 2. By definition, the non-dispersive phase velocity is a constant. However, the dispersive phase velocity decreases non-linearly as the wave number increases. Moreover, the group velocity further decreases non-linearly as the wave number increases. Importantly, with dispersion, all three behave fundamentally differently. This, in fact, is the manifestation of dispersion. Without dispersion, all three would be the same with a constant value, the non-dispersive phase velocity.

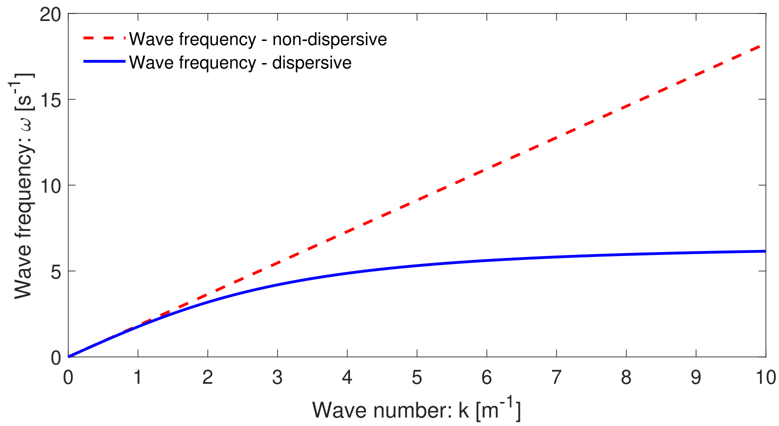

Fig. 1. The wave frequency as a function of wave number given by (10). The wave frequency without and with dispersion are fundamentally different, and differ largely for higher wave number.

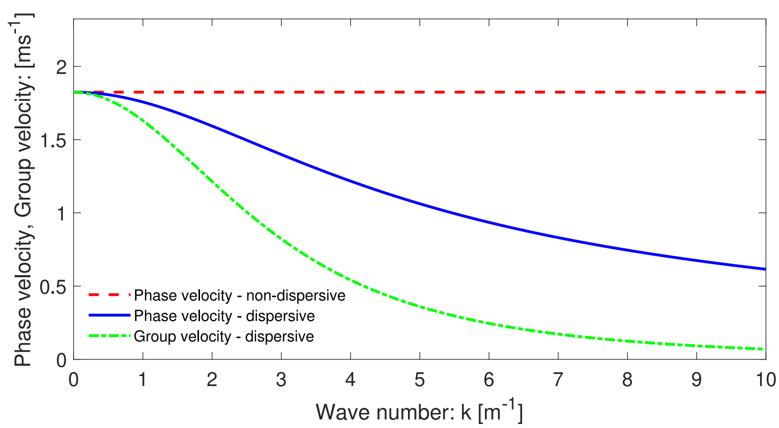

Fig. 2. The phase velocity and group velocity as functions of the wave number given by (10) and (12). The non-dispersive phase velocity is constant. The dispersive phase velocity and group velocity decrease non-linearly with the wave number. The dispersive group velocity is the lowest among the three.

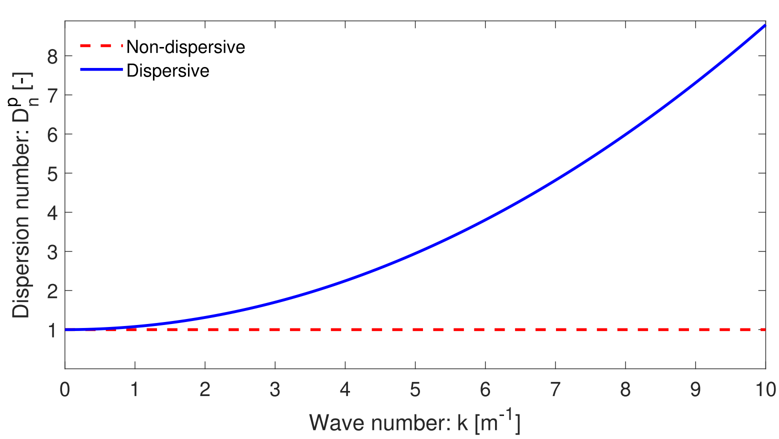

Fig. 3. The dispersion number $\mathcal{D}_n^p$ as a function of the wave number k given by (13). Also shown is the reference when the dispersion is absent.

4.3. The dispersion number

The dispersion number $\mathcal{D}_n^p$ given in (14) is presented in Fig. 3. It shows that the dispersion number increases rapidly as the wave number increases. This resulted due to the stretching of $\mathcal{D}_n^p$ as given in (13), and summarizes the overall effect of dispersion in the wave dynamics in mass flow.

4.4. Influence of parameters

The wave frequency, phase and group velocities, and the dispersion number, ω, C_p, C_g and $\mathcal{D}_n^p$, all depend collectively on the effective lateral stress $\mathcal{T}$ and the dispersion parameter $\mathcal{D}$. However, explicitly, they depend either linearly or non linearly on the solid volume fraction α_s, the earth pressure coefficient K, buoyancy or lubrication effect γ,

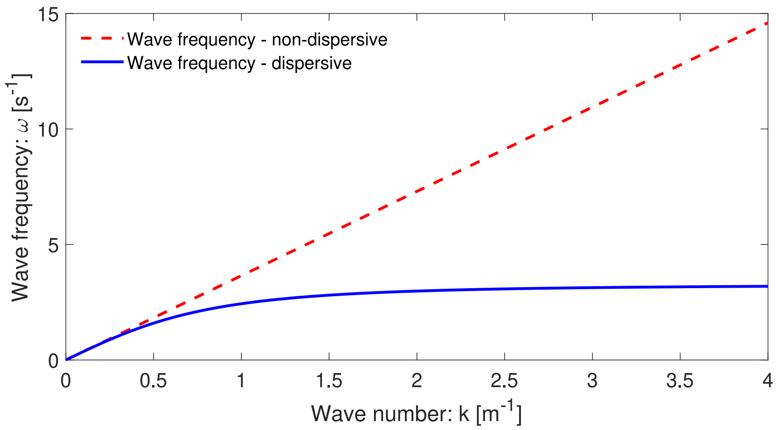

Fig. 4. The wave frequency as a function of wave number as in Fig. 1, but now with $H = 2$. The wave frequency without and with dispersion are fundamentally different, and differ largely for higher wave number, more than in Fig. 1.

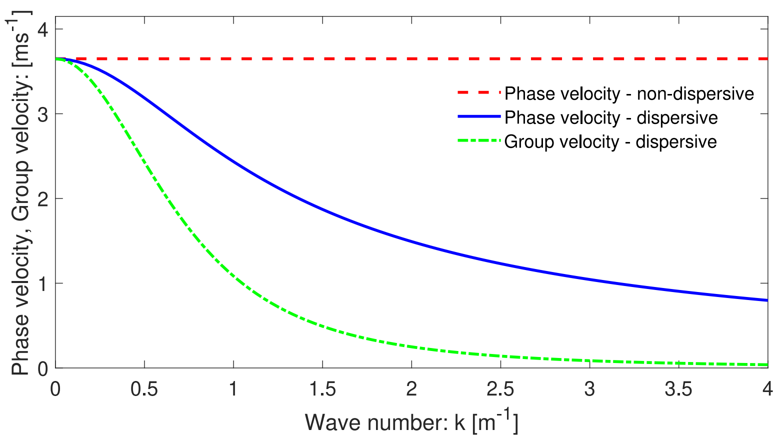

Fig. 5. The phase velocity and group velocity as functions of wave number as in Fig. 2, but now with $H = 2$. The non-dispersive phase velocity is constant. The dispersive phase velocity and group velocity decrease non-linearly with the wave number, faster than in Fig. 2. The dispersive group velocity is the lowest among the three.

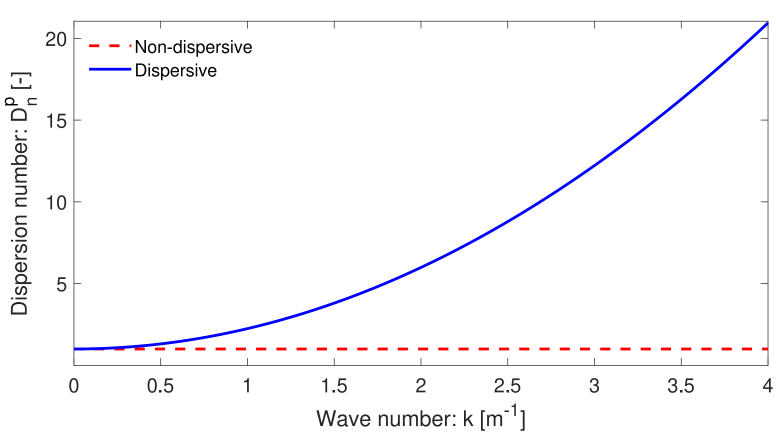

Fig. 6. The dispersion number $\mathcal{D}_n^p$ as a function of the wave number k as in Fig. 3, but now with $H = 2$, which now increases more rapidly than in Fig. 3. Also shown is the reference when the dispersion is absent.

the channel slope ζ, and the mean material depth H. The detailed analysis can be carried out based on all these parameters. Particularly important are α_s, K and γ as they carry crucial physical information of the solid particles and the fluid in the mixture. Nevertheless, as seen from the representations and definitions of ω, C_p, C_g and $\mathcal{D}_n^p$, H plays a rather key role in determining the wave frequency, phase and group velocities, and the dispersion number, because $\mathcal{D}$ varies quadratically with H. So, here, I only focus on H. I increase its value from 0.5 to 2. The results are presented in Figs. 4, 5 and 6 for the wave frequency, the phase and group velocities and the dispersion number, respectively. Comparing these figures with their counterparts, Figs. 1, 2 and 3, it is evident that the wave frequency, phase and group velocities all decrease strongly with the increased H value and saturate much earlier with their lower values within the domain of smaller wave number k. However, the dispersion number increases rapidly with the increased H value. This is also what the structures of these variables tell us from their analytical representations, because all of ω, C_p, and C_g are somehow inversely related with $\mathcal{D}$, but, $\mathcal{D}_n^p$ is linearly related with $\mathcal{D}$.

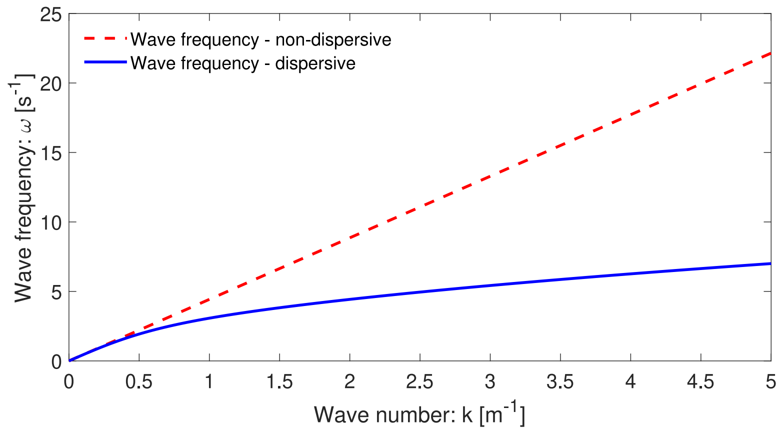

Fig. 7. The wave frequency as a function of wave number for surface water waves for intermediate water depth.

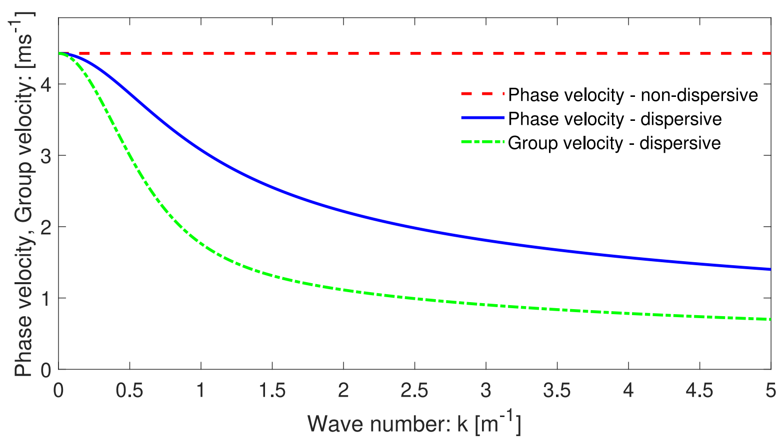

Fig. 8. The phase velocity and group velocity as functions of wave number for surface water waves for intermediate water depth.

Yet, note that the non-dispersive phase velocity is now significantly higher than in the previous figure, and the rate at which the phase and group velocities decrease is much higher than the same in the previous figure. This also resulted in the rapid increase of dispersion number than in the previous figure for higher mean material depth.

4.5. Comparison with the surface water wave

The classical shallow water surface waves are non-dispersive. For deep water surface waves, the phase velocity is $C_p = \sqrt{g/k}$ and the group velocity is one half of the phase velocity, $C_g = 0.5C_p$ and are independent of the fluid depth. This is not relevant for us for the present consideration. So, with respect to the dispersion relation, the intermediate fluid depth is relevant here. The phase velocity for the intermediate water depth is $C_p = \sqrt{\frac{g}{k}\tanh(Hk)}$, while the group velocity is $C_g = \frac{1}{2}\left[1 + \frac{2Hk}{\sinh(2Hk)}\right] C_p$, see, e.g., [21]. Dispersion relations for the water waves with the intermediate depth are presented in Fig. 7 for the wave frequency, in Fig. 8 for the phase and group velocity, and in Fig. 9 for the wave number, respectively, with the fluid depth $H = 2$ (chosen this way for the comparison reason). Compared with the corresponding figures Figs. 4, 5 and 6, we observe that the new dispersion relations derived in Section 3 behave fundamentally differently than the dispersion relations for the water waves with depth. Particularly interesting is the dispersion number. While the new dispersion number derived here for mass flow increases continuously as a quadratic function of the wave number with its minimum value of unity, the dispersion number for the water waves also begins at its minimum value of unity, then hyperly rises up, but then asymptotically approaches the dispersion number (two) of the deep water waves already at about (relatively low) wave number $k \geq 2$.

5. Discussion

5.1. General aspects of the landslide dispersion relation

We observed the following important physical phenomena from the landslide dispersion relation presented in Section 3.

- As C_p depends on the wave number k, the wave under consideration is strongly dispersive.
- The new dispersion relation includes many different physical parameters and mechanical responses.
- The usual shallow water wave ($\alpha_s = 0, K = 1$, and the term with $\mathcal{D}$ can be neglected) is a special case: $C_p^w = \sqrt{g^z H}$.
- Classical debris-avalanche motion is a special case when the non-hydrostatic contribution is neglected, i.e., $C_p^d = \sqrt{\mathcal{T}}$, which is the wave speed, and can be written in alternative form as:

$$C_p^d = \sqrt{\mathcal{T}} = \sqrt{\left[\left((1-\gamma)K+\gamma\right)\alpha_s + \left(1-\alpha_s\right)\right] g^z H}$$
$$= \sqrt{\frac{1}{\rho}\left[\left((1-\gamma)K+\gamma\right)\alpha_s + \left(1-\alpha_s\right)\right]\rho g^z H} = \sqrt{\frac{T}{\rho}}, \quad (15)$$

where, ρ is the mixture bulk density and $T = \left[\left((1-\gamma)K+\gamma\right)\alpha_s + \left(1-\alpha_s\right)\right]\rho g^z H$ is the lateral stress (compressive for mass flow). So, without the dispersion (non-hydrostatic) contribution, the square of the phase velocity is the ratio between the lateral compressive stress and the material density. Which is similar to the phase velocity for a string in which the ratio is between the tension and the density. Thus, for the mass flow, the compressive stress involves gravity, particle concentration, buoyancy, the earth pressure coefficient and the mean depth of the debris

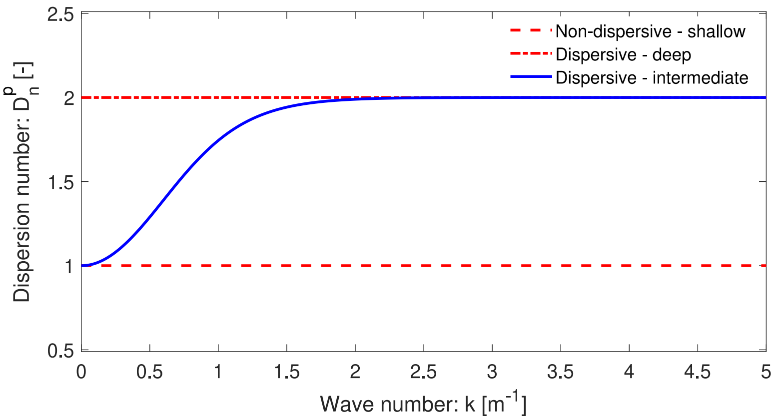

Fig. 9. The dispersion number as a function of the wave number for surface water waves for intermediate water depth.

material, which for the classical shallow water is only related to gravity and the mean water depth. There are other important aspects: Phase velocity is high for compressional flows (for which $K > 1$), and low for dilational flows ($K < 1$). Similarly, phase velocity is high for pure granular flow ($\alpha_s = 1, \gamma = 0$), and reduces for the particle fluid mixture flows, with its minimum for fully buoyant flows or when the particle concentration vanishes, turning it in to the pure fluid flow. These physical mechanisms are consistent with the strength of material.

- Classical shallow water and debris flow models are non-dispersive.
- **Dispersive lateral stress and dispersion intensity:** Consider C_p in (10) in its full form involving $\mathcal{D}$:

$$C_p = \sqrt{\frac{\mathcal{T}}{1+\mathcal{D}k^2}} = \sqrt{\frac{\left[\left((1-\gamma)K+\gamma\right)\alpha_s + \left(1-\alpha_s\right)\right]\rho g^z H}{\left(1+\mathcal{D}k^2\right)\rho}}$$
$$= \sqrt{\frac{T}{\left(1+\mathcal{D}k^2\right)}\frac{1}{\rho}} = \sqrt{\frac{T_e}{\rho}}, \tag{16}$$

where $T_e = T_e(k) = T/\left(1+\mathcal{D}k^2\right)$. We call T_e the effective dispersive lateral stress. This means that as the wave number increases, the effective dispersive stress decreases, however without changing the material density. This ultimately decreases the phase velocity, as in the reduced phase speed in string, but, with less tension. Importantly, the dispersion relation emerges due to the dispersion parameter $\mathcal{D}$ which varies linearly with the solid particle concentration α_s and the lateral pressure coefficient K, and quadratically with the depth H. So, the dispersion intensity increases linearly with α_s and K, and quadratically with H and k. Therefore, dispersion is strong for relatively thick flows, and for large wave number.
- **The anti-restoring force in landslide:** $\mathcal{D}$ in the denominator in C_p, i.e., in $\left(1+\mathcal{D}k^2\right)$, generates the dispersive wave. This, in our consideration, originates from the acceleration of the debris material in the slope normal direction (including drags and virtual mass forces in real mixture where the relative acceleration between particle and fluid is not negligible) in excess to the hydrostatic force (the material load). In (16), the restoring force is decreasing as a function of the wave number together with the dispersion parameter. So, C_p for landslide induces an anti-restoring force. For debris material during the primarily down-slope motion this contributes positively, because this is the anti-restoring force. This is in contrast to the classical dispersive wave in string with stiffness, which is the restoring force. Due to the anti-restoring force, landslides are more dispersive than the piano strings. This reveals that the dispersion behavior in mass flow is fundamentally different than that in classical stiff-string wave motion. However, my dispersion relation, in principle, agrees with classical water waves: waves with higher wave length move faster.
- All the frequencies and modes of dispersion of waves in landslides can be acquired from (6) or (8) from which we may construct the sounds associated with landslides as for piano.
- For a reasonably larger wave length the surface tension effect can be neglected. And thus, the gravity-capillary wave can well be approximated simply by the surface-gravity wave. For this reason, I have neglected the surface tension.
- As friction and slope geometry are other important aspects in mass flows, the more complete picture of the wave dispersion in landslide can be achieved by including the additional effects of the friction and topography (curvature) related terms in (6). This may result in a complex combination of restoring- and anti-restoring force regimes, possibly with the group velocity being in the direction opposite to the phase velocity. These sophisticated aspects can be dealt with separately.

Important aspect here is the derivation of the complex dispersive partial differential equation for landslide which reduces to the simple classical wave equation. The new dispersion relations behave fundamentally differently than the dispersion relations for the water waves as the usual shallow water wave is a special case. The landslide dispersion relation is perceptibly explained as it incorporates several physical parameters and mechanical responses. It establishes a crucial relationship between the phase velocity and the group velocity. Importantly, the wave dispersion for landslide appeared to be different than the wave dispersion in the piano string which reveals the fact that landslides can be more dispersive than the piano strings.

5.2. Implications of the dispersion relation in mass flow simulations

The above results demonstrate the importance of dispersion in legitimately simulating the wave phenomena in naturally dispersive mass flows. The very special form of the wave frequencies, phase and group velocities and the dispersion number shown in Figs. 1, 2 and 3 are due to the novel dispersive wave Eq. (6), or the dispersion relation (10), representing the mass flow problem incorporating the effective lateral stress (normalized by mass density) $\mathcal{T}$, and the dispersion parameter $\mathcal{D}$. The overall wave dynamics are determined by $\mathcal{T}$ and $\mathcal{D}$, while dispersion is solely dependent on $\mathcal{D}$. The major feature of the dispersion relation is to tell us how the waves of different wave lengths move with different frequencies. So, it can play an important role in debris surge generation and attenuation.

6. Summary

Based on the non-hydrostatic mass flow model [1], I derived a novel dispersive wave equation, or a dispersive partial differential equation, the first of this kind, for the landslide motion. This reduces to the simple classical wave equation when the non-hydrostatic dispersion effects are ignored. The new system of dispersive wave for debris mixture recovers the classical dispersive water waves as a special case. The frequency dispersion relation to my model for mass flow is different than the classical linear frequency dispersion for Boussinesq water wave equations. Obtained results show that the wave frequency with dispersion increases non-linearly as a function of the wave number. The wave frequency without and with dispersion are fundamentally different. For dispersive landslides, the wave frequency without dispersion appears to heavily overestimate the dispersive wave frequency for higher wave number. Due to the dispersion parameter emerging from the non-hydrostatic contribution for mass flow, the phase velocity becomes a function of the wave number. This gives rise to the group velocity that is significantly different from the phase velocity, characterizing the dispersive mass flows. The dispersive phase velocity and group velocity decrease non-linearly with the wave number. The dispersive group velocity is substantially lower than the phase velocity.

I analytically derived the dispersion number as the ratio between the phase velocity and the group velocity. The dispersion number measures the deviation of the group velocity from the phase velocity and provides a dynamic scaling between these two velocities. The dispersion number increases rapidly as the wave number increases, and summarizes the overall effect of dispersion in the wave dynamics in mass flow. While the dispersion number for mass flow increases continuously as a quadratic function of the wave number, the dispersion number for the water waves (for intermediate depth) is strongly bounded (within the small wave number) between the shallow water and the deep water dispersion numbers. Along with other physical parameters, the mean flow depth plays an important role in determining the wave frequency, phase and group velocities, and the dispersion number. My model and results demonstrate the importance of dispersion in legitimately describing the wave phenomena in dispersive mass flows.

I defined the effective dispersive lateral stress for landslide. As the wave number increases, the effective stress decreases, however without changing the material density. Contrary to the classical dispersive wave in string with stiffness, which is associated with the restoring force, I proved the existence of an anti-restoring force in landslide. Due to the anti-restoring force, landslides are more dispersive than the piano strings. This reveals the fact that the wave dispersion in landslide is fundamentally different than the wave dispersion in the piano string. As for piano, there is now a possibility to construct the sounds associated with dispersive landslides.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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