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Mathematical Model and Experimental Validation of Free Surface Size Segregation for Polydisperse Granular Materials

Shiva P. Pudasaini, Jan Mohring

Abstract A mathematical model of free surface size segregation of granular matter is proposed. The material is assumed to be dry, cohesionless and to consist of several populations with distinct diameter ranges. It is not necessary that one population is predominant. The extent of segregation depends mainly on the relative amount of small and large particles and on the diameter ratios but not on their absolute values. The model predicts how the fractions of these populations change from the position where the material is fed onto a heap to its bottom. It is based on ideas of L. PRIGOZHIN to treat binary mixtures. Introducing a new scheme to interpolate the local deposition rates from situations with one predominant population it becomes possible to handle mixtures with three and more diameters. The model may be applied to storage systems of any kind, such as bunkers, hoppers, silos or mixing beds, provided the source is sufficiently weak and concentrated. Numerical simulations are discussed for conical and ramp-like geometries and for ternary mixtures. In order to validate the model a few experiments were conducted. As measured by the natural variability of the problem and the simplicity of the model, which is sufficiently easy to be used in an industrial controller, good agreement between theory and experiment is found.

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S. P. Pudasaini (✉)
Department of Applied Mechanics,
Institute for Mechanics-AG III,
Darmstadt University of Technology,
Hochschulstrasse 1, D-64289 Darmstadt, Germany
e-mail: pudasain@mechanik.tu-darmstadt.de

J. Mohring
Fraunhofer Institut Techno- und Wirtschaftsmathematik,
Erwin-Schrödinger-Strasse,
D-67663 Kaiserslautern, Germany
e-mail: mohring@itwm.fhg.de

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Keywords Free surface flow, Size segregation, Polydisperse granular material, Bulk solid, Ternary mixture

1 Introduction

If a granular material is poured e.g. into a silo from a small *point-source* at the center, a conical pile is formed [3]. The material is continuously deposited at the top of the heap and flows down the faces immediately in a thin layer. In this layer of grains, *free surface size segregation* takes place, and the extent of segregation depends mainly on the *size ratio* and *relative number* of small and large particles [2]. The grains considered here differ only in size but not in other properties like density, shape and roughness of the surface. Surface size segregation is observed in many industrial applications as well as in the geophysical contexts [6], [9], [15]. Since segregation is chiefly responsible for the variation of the quality of the *bulk solid*, for structural design of bunkers, hoppers and silos one should take these effects into account [2]. Once in motion kinetic sieving of a *polydisperse granular mixture* creates small shear bands in which the larger particles overlie the smaller particles and *kinetic sieve* sorts the granular material by grain size [3], [16], [17]. This hypotheses says that small grains fall into the gaps between large grains. When a *dry cohesionless* granular material is discharged onto a free surface, particles differing by their size become separated on the heap. Under the action of gravity the smaller particles are more likely to fall into gaps that open up beneath them than the large grains, because they are more likely to fit into the space available [3]. Consequently, small particles tend to stay close to the point of discharge and larger particles move toward the far end of the heap.

When a heap is formed by pouring particulate materials onto it two main processes determine formation of a filling: pouring particles down the free surface of the heap and flow of material within the body [13]. Control of the space distribution of the different species in the heap of non-homogeneous bulk solids is essential in order to reduce the composition variability of discharged or reclaimed materials. Both the inner and *free surface flow* of the bulk solid influence such distributions and cause segregation. Although a vast literature is concerned with the gravity flow of granular materials in bunkers and different vessels [3], less attention has been paid to bulk solid flows down heap slopes [18]. Such a flow is mainly confined to a very thin boundary layer and does not penetrate into the bulk of the material [10]. The surface flow is the main agent controlling the evolution of the heap on the given rough

support surface under a source of bulk substance [13]. Determination of the form of such a heap is of great interest for metallurgy, geomorphology, pyroclastic flow and turbidity current depositions [4], [5], [20]. Knowledge of the form of the free boundary is quite necessary mainly to evaluate the free surface size segregation, which in most process engineering scenarios and geophysical contexts is much more significant than any other kind of segregation.

A quantitative study of segregation in granular materials based on experimental data of Drahun & Bridgwater [2] is presented by Prigozhin [13] who proposed a continuum model depending on the diameter ratio and the concentrations of the particles in the granular mixture. This model, which is however applicable only to binary mixtures, features a mathematical tool, namely *Variational Inequality* [12]. To the best of our knowledge, up to now the only quantitative study of free surface size segregation, both in experiment and theory, is available only for *binary mixtures* [2], [13]. To capture the effects of size segregation in industrial applications or in natural events, the limitation to binary mixture is too severe. This fundamental problem is actually a pressing subject of investigation and still a challenge for mathematicians, physicists, chemists and engineers. Indeed, applications may require models with continuum or at least polydisperse size distribution of the grains. Here, an improved and extended mathematical model is presented which is able to give the answer to the question of size segregation on the free surface of a granular heap made of two or more components. Different heap geometries are also accounted for. For binary mixtures, the model reduces to a special case of Prigozhin's model. Comparison of the extended model applied to binary mixtures with Prigozhin's model shows that the new model is as accurate as but much simpler than the previous model. For *ternary mixtures* the new model is able to reflect the qualitative as well as quantitative behaviour of free surface size segregation. Experiments were conducted similarly to that of Drahun and Bridgwater [2], and good agreement between theory and experiment is observed.

The organisation of the paper is as follows: In the second section a brief review of Prigozhin's model for free surface size segregation is presented. In the third section a new mathematical model is developed to extend Prigozhin's model, and numerical simulations are presented for different heap geometries. Section four deals with the comparison of Prigozhin's model with our new model for the experimental data of Drahun and Bridgwater. The experimental set up, measured data and related concepts are discussed in section five. Comparison between the new and extended mathematical model and the experimental data is presented in section six. Finally, conclusions are contained in section seven.

2 Prigozhin's Mathematical Model for Free Surface Size Segregation

2.1 Evolution of the granular body

Prigozhin has proposed a continuum model for the quantitative study of free surface size segregation of dry-cohesionless bidisperse granular materials mainly depending on the material concentrations of the different constituents in the bulk and their diameter ratios that essentially lie in the range (0.4, 1.6). This will be clear in the sequel. Prigozhin applies the following assumptions for his model: (a) discharged material flows along the surface, in the direction of steepest descent, if the angle of the heap exceeds the angle of repose; (b) granules are poured from a quasi-stationary weak source, the material is distributed infinitely fast in arbitrarily thin layers; (c) constant angle of repose; (d) the source material constitutes a binary mixture.

Here we will give a brief review of Prigozhin's model [13]. Let $\mathbf{\Omega} \subset \mathbb{R}^2$ be bounded by vertical walls, $y = g(x)$ the equation of the rigid support surface, $y = h(x,t)$ the equation of the free surface of the body, ξ the parameter determining the type of material in a mixture and $\kappa(\xi, x, y, t)$ the *concentration* (density of mass distribution) of this material, see Fig. 1. The material distribution must satisfy the normalisation condition $\kappa \geq 0, \int \kappa d\xi = 1$.

Consider the vertical projection $U(x,t)$ of the total surface mass flux. Let $\kappa_+(\xi, x, t)$ be the concentration of material ξ *in this flux*. Analogously, let $\kappa_-(\xi, x, t)$ be the composition of the heap surface layer determining conditions of flow and sedimentation. Due to normalisation we have

$$\kappa_+ \geq 0, \int \kappa_+ d\xi = 1 \ . \tag{1}$$

Moreover, let

$$\dot{h} = \frac{\partial h}{\partial t} + v_x \nabla h - v_y \ ,$$

be the total derivative of h with respect to time and v_x, v_y the components of the velocity $v(x, h(x,t), t)$. This means that $\dot{h} = 0$ is the kinematic equation of the free surface of the heap $h(x,t) - y = 0$. Notice that for scalar x, $\nabla h = \partial h / \partial x$. Let $\omega(\xi, x, t)$ be the intensity of the external source, $\kappa_s(\xi, x, t)$ the material concentration of material ξ *in the heap surface layer* which is being generated ($\dot{h} > 0$)

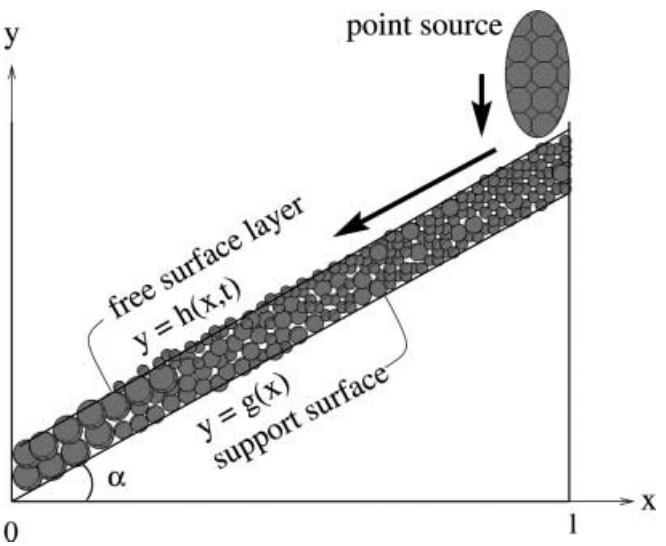

Fig. 1. Schematic diagram of the apparatus of Drahun and Bridgwater and particle segregation by their size

or poured out ($\dot{h} < 0$) and ρ the bulk density of the granular material with mass fraction κ_s. Now we consider the flow of particles pouring down the free surface of the heap in a thin layer. If there is no percolation of small particles from the surface into lower heap layers, the equation of material balance for any type of material in a mixture can be written as (we refer to [1], [19])

$$\kappa_s \rho \dot{h} + \nabla(\kappa_+ U) = \omega \ . \tag{2}$$

If the kinetic energy of the flowing particles is small and rapidly lost in collisions, we can neglect the inertia and assume that the material flowing over the free surface is directed towards the steepest descent, implying

$$U = -m\nabla h \ , \tag{3}$$

where $m \geq 0$ is an unknown *diffusion coefficient*, which corresponds to the variational principal of PRIGOZHIN, is determined by the fact that the surface never becomes steeper than the angle of repose. The balance equation of mass for constituent ξ then assumes the form

$$\kappa_s \rho \dot{h} - \nabla(\kappa_+ m \nabla h) = \omega \ . \tag{4}$$

Integrating equation (4) with respect to ξ and using the normalisation conditions for κ_s and κ_+, we obtain

$$\rho \dot{h} - \nabla(m \nabla h) = \omega_\Sigma \ , \tag{5}$$

where $\omega_\Sigma = \int \omega d\xi$. The free surface of the granular body can then be found from (5) if m is known without determining the distribution of the material composition, for more details see [13], [14].

2.2
Composition of new surface layer when segregation is absent

Let for the moment all particles have the same size, but distinguish the different constituents by colours. To determine the space distribution of the differently coloured (identical) material in a filling, it is first necessary to find the diffusion coefficient $m(x,t)$ as a solution of (5) in which $h(x,t)$ is a known function. For this we also need the condition that the surface of the granular body never becomes steeper than the angle of repose of the bulk. In this paper, we will consider only the growth of the surface and we will neglect the destruction of former layers. That is, we always take $\dot{h} > 0$. The concentrations κ_s and κ_+ can then be found from equation (4) and the relation

$$\kappa_s = \kappa_+ \tag{6}$$

holds true in the absence of segregation (assuming that h and m are known). The realization of this process is not difficult for problems with *plane* ($k = 0$) or *axisymmetric* ($k = 1$) of the heap geometries. To confine the general equation of mass balance (5) for a plane and axisymmetric geometry and to write it in a compact form we replace x by r so that we can rewrite this equation as

$$-\frac{1}{r^k}\frac{\partial}{\partial r}\left(r^k m \frac{\partial h}{\partial r}\right) = \omega_\Sigma - \rho \dot{h} \ . \tag{7}$$

However, for notational convenience we will replace r by x when dealing with the flow of mass on an inclined plane and r will be kept unchanged for the conical pile in the sequel. Equation (4) is now written as

$$-\frac{1}{r^k}\frac{\partial}{\partial r}\left(\kappa_+ r^k m \frac{\partial h}{\partial r}\right) = \omega - \kappa_s \rho \dot{h} \ , \tag{8}$$

and formula (6) may be used to determine the composition of materials in the surface flow. For an inclined plane and a cone the shapes of the surface are known to be a plane and a cone, respectively and m can be determined. This will be clear from §2.3 and §3.4.

2.3
Free surface segregation for the flow in an inclined plane

Under the assumptions of small discharge rates, slow heap growth and flow of bulk materials over the free surface in thin well-mixed layers, the local composition κ_s depends on

1. the local composition κ_+ of the surface flow, since the flowing material is used for new layer formation,
2. the local composition κ_- of the heap surface layer determining conditions of the flow and sedimentation.

Essentially the same algorithm of §2.2 allows determination of new surface layers when segregation occurs but, in this case, relation (6) should be modified by excluding κ_- from considerations. We obtain a modified form of equation (6) valid when segregation occurs, namely

$$\kappa_s = S(\kappa_+) \ , \tag{9}$$

where S is an operator determined by the mechanism of particle segregation. S does not depend on the heap geometry or source distribution. If this operator were known, the distribution of the material composition could be found from equations (7), (8), and (9). For the determination of this operator PRIGOZHIN uses experimental data [2]. The composition of a binary system can be determined by the concentration of one of the components. Referring from now on to κ_+ and κ_s as concentrations of this chosen component, we can consider S to be a function of κ_+ rather than an operator which maps $(\kappa_+, 1-\kappa_+)$ onto $(S(\kappa_+), 1-S(\kappa_+))$. The component present in small quantities, the weight of which does not exceed 1% of the bulk material, will be called *tracer* and the other component will be called *bulk*. In this paper the concentration of the component whose weight does not exceed 1% will also be called *low concentration* and the concentration of the component whose weight exceeds 1% will be called *high concentration*. The geometrical part of this problem with the point source on the right boundary, see Fig. 1, where α is the angle of repose of the bulk material and l the length of the horizontal projection of the inclined plane on which the granular flow occurs, (i.e. the length of the bunker [2]) has an obvious solution- the flat surface of filling moves upwards with velocity

$$\dot{h} \equiv \frac{\partial h}{\partial t} = \frac{1}{\rho l}\frac{dM}{dt} \ ,$$

maintaining its slope at the angle of repose. Here dM/dt is the flow rate of mass from the source. The point source of the material is positioned just above the right top of the

inclined plane. This means that $\omega = \frac{dM}{dt}\delta(x-l)$, where δ is the *Dirac* δ-functional. Hence, for $k=0$ and integrating equation (7) yields

$$m\frac{\partial h}{\partial x} = \frac{x}{l}\frac{dM}{dt} \ . \tag{10}$$

Everywhere on the surface $\kappa_s = S(\kappa_+)$ since only the growth of surface layers occurs in this case. Equation (8) has a δ-function on its right-hand side and is equivalent to the following problem: Plugging (10) into (8), using that $dM/dt =$ constant and setting $k=0$ yields

$$x\frac{d\kappa_+}{dx} = S(\kappa_+) - \kappa_+, \quad \kappa_+(l) = \kappa_0 \ , \tag{11}$$

where κ_0 is the concentration of the species in the fed material and the corresponding points $x=0$ and $x=l$ are shown in Fig. 1. When there is (almost) no material of a certain kind in the flux of mass, (almost) no amount of material of that kind is deposited in the new layer. Taking this fact into account i.e. $S(0)=0$ and using a Taylor expansion for the tracer component, the unknown function S near $\kappa_+ = 0$ can be approximated by a linear function $S = C\kappa_+$, where C is a constant. Hence equation (11) reduces to

$$x\frac{d\kappa_+}{dx} = (C-1)\kappa_+, \quad \kappa_+(l) = \kappa_0 \ . \tag{12}$$

Solving this equation and calculating the tracer concentration of the material ξ in the flux, one can find the density of spatial distribution of the tracer to be

$$q(x) := \frac{\kappa_s(x)}{\int_0^l \kappa_s(x)dx} = \frac{C}{l}\left(\frac{x}{l}\right)^{C-1} \ . \tag{13}$$

Determination of C for various diameter ratios d_t/d_b of tracer and bulk material may be based on comparison of the theoretical mean tracer position

$$\frac{\bar{x}}{l} := \frac{1}{l}\int_0^l xq(x)dx = \frac{C}{C+1} \ , \tag{14}$$

with experiments. For small tracer concentrations this mean value depends linearly on the diameter ratio with experimental dependencies given by Drahun and Bridgwater [2]. One of the most remarkable facts found by these authors is that this dependence is not affected by the type of material and can be presented as

$$\frac{\bar{x}}{l} \sim \left[0.5 - 0.83\left(\frac{d_t}{d_b} - 1\right)\right] \ , \tag{15}$$

for $0.4 < d_t/d_b < 1.6$. When the size ratio is less than about 0.4, *almost all* tracers are concentrated just below the source ($\bar{x} = l \Longrightarrow C = \infty$) and when the size ratio is larger than 1.6, *almost all* tracers are concentrated at the opposite end of the inclined (plane) surface ($\bar{x} = 0 \Longrightarrow C = 0$). From these limiting behaviours and (14) and (15) we can easily derive the following formula to determine the parameter C reflecting the mechanism of segregation on the freely growing heap surface:

$$C\left(\frac{d_t}{d_b}\right) = \begin{cases} \infty & \text{if } \frac{d_t}{d_b} \le 0.4, \\ \frac{1.6-\frac{d_t}{d_b}}{\frac{d_t}{d_b}-0.4} & \text{if } 0.4 < \frac{d_t}{d_b} < 1.6, \\ 0 & \text{if } \frac{d_t}{d_b} \ge 1.6 \ . \end{cases} \tag{16}$$

With this value of C, the problem of finding the density of the spatial distribution of tracer material in the binary mixture can be solved using equation (13). According to the arguments given by Drahun and Bridgwater and Prigozhin this model is valid whenever $d_t/d_b \in (0.4, 1.6)$. Now we wish to find an approximation of the function S for higher concentrations κ_+. Let us denote the components of the binary system by indices 1 and 2, S being the unknown relation between the concentrations (κ_+ and κ_s) of the first component. From the previous discussion, it is known that in the neighbourhood of $\kappa_+ = 0$ the first material is the tracer for which

$$S(0) = 0 \quad \text{and} \quad \frac{dS}{d\kappa_+}(0) = C_1 \ , \tag{17}$$

where $C_1 = C\left(\frac{d_1}{d_2}\right)$. Similarly, in the neighbourhood of $\kappa_+ = 1$ the second material is the tracer for which

$$S(1) = 1 \quad \text{and} \quad \frac{dS}{d\kappa_+}(1) = C_2 \ , \tag{18}$$

where $C_2 = C\left(\frac{d_2}{d_1}\right)$. It is reasonable to suppose that S is a smooth monotonous function. Let κ_+^1 and κ_+^2 be the concentrations of the first and second constituents in the flux. We choose them as two independent axes of the Cartesian coordinate system, see Fig. 2, however since $\kappa_+^1 + \kappa_+^2 = 1$ possible values of $\kappa_+^{1,2}$ are only along the straight line connecting the points A and B in Fig. 2. So, S may be plotted perpendicularly to this κ_+-axis (the line AB) with boundary values and their slopes given in A and B as indicated. Prigozhin uses the following two parabolic functions to determine the S function

$$\kappa_+(\tau) = a_1\tau^2 + a_2\tau + a_3, \quad \kappa_s(\tau) = b_1\tau^2 + b_2\tau + b_3 \ , \tag{19}$$

where $\tau \in [0,1]$ is a parameter and $\kappa_+(0) = 0$ and $\kappa_+(1) = 1$. Determining the coefficients from the above boundary conditions, (17) and (18), the function S of equation (9) has the following parametric representation:

$$\begin{aligned} \kappa_+(\tau) &= \tau\left[2(C_2-1) + (2-C_1-C_2)\tau\right]/(C_2-C_1), \\ \kappa_s(\tau) &= \tau\left[2C_1(C_2-1) + (C_1+C_2-2C_1C_2)\tau\right]/(C_2-C_1) \ . \end{aligned} \tag{20}$$

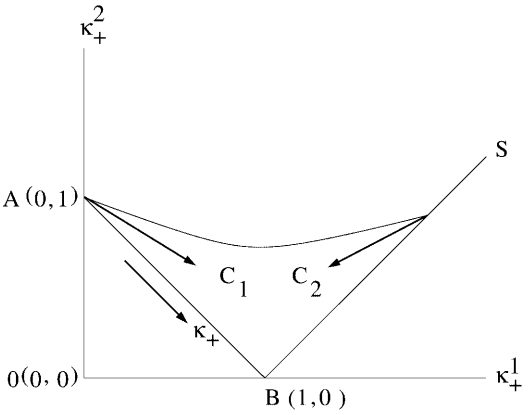

Fig. 2. Interpolation giving the function S

Note that (20) is just a simple function which satisfies the boundary conditions and is monotonous. There is no other mathematical or physical reason to choose this parameterisation and in §3 we will present an alternative which is not only as good as this but can be extended to more than two constituents. Equation (20) has four independent parameters, say $a_1^*,\ a_2^*,\ b_1^*,\ b_2^*$, and can be rewritten as

$$\kappa_+(\tau) = \tau\,[a_1^* + a_2^*\tau]\,/\,[a_1^* + a_2^*],$$
$$\kappa_s(\tau) = \tau\,[b_1^* + b_2^*\tau]\,/\,[a_1^* + a_2^*]\ ,$$

where the a_i^* and b_i^* are easily identifiable from (20). We will show in §3 that it can be substituted by taking an *algebraic* function with four parameters so that we have an equal number of unknowns and boundary conditions to determine the function S. This means that such a coupling of κ_+ and κ_s, as used by PRIGOZHIN, is not necessary. From (11) and (20), using τ as a new dependent variable, the following *initial value problem* is obtained:

$$x\,[(C_2-1)(1-\tau) - \tau(C_1-1)]\,\frac{d\tau}{dx} = \tau(1-\tau)(C_1-1)(C_2-1), \quad \tau(l) = \tau_0\ ,$$

where τ_0 is the only root of the second order equation $\kappa_+(\tau_0) = \kappa_0$ in the interval $(0,1)$. An analytical solution of this equation can be found in implicit form

$$\left(\frac{\tau}{\tau_0}\right)^{\frac{1}{C_1-1}} \left(\frac{1-\tau}{1-\tau_0}\right)^{\frac{1}{C_2-1}} = \frac{x}{l}\ . \tag{21}$$

Finally, (20) and (21) are used to obtain numerically the desired function $q(x)$ in (13).

3
Extension of Prigozhin's Model to Ternary Mixtures

3.1
Need for extension

PRIGOZHIN'S model gives the answer to the question of size segregation for a material composite of two components only. However, in industrial applications or in natural phenomena a continuum model is needed that provides a quantitative answer to the size segregation on the free surface of a granular material that consists of two or more components. In this section, we will present a new model for size segregation for three components, which can easily be extended to more than three components [14].

3.2
The operator S

The following notation will be employed, i running always from 1 to 3:

- κ_+^i : concentration of the material i in the flux,
- S^i : component i of operator S,
- κ_s^i : concentration of the material i in the new surface layer,
- q^i : density of the spatial distribution of the material i.

Here i corresponds to ξ of §2.1. The main problem is to find the unknown operator S determined by the mechanism of particle segregation in an appropriate manner. We wish to develop a model for three components for which $0 \le \kappa_+^1, \kappa_+^2, \kappa_+^3 \le 1$ and $\kappa_+^1 + \kappa_+^2 + \kappa_+^3 = 1$. Consider the three dimensional rectangular Cartesian coordinate system with coordinates $\kappa_+^1,\ \kappa_+^2$ and κ_+^3 as shown in Fig. 3. The generalisation of the construction of Fig. 2 is then the equilateral triangle of Fig. 3, and permissible κ_+-values lie on the triangle with the edges $(1,0,0), (0,1,0), (0,0,1)$, as indicated in Fig. 3. This will be called the "base plane". Let $\hat\kappa_+^1,\ \hat\kappa_+^2$ and $\hat\kappa_+^3$ be the unit vectors along the respective coordinate axes. Any point on this base plane is readily seen to be representable by the equation $\kappa_+ = \hat\kappa_+^1 + \kappa_+^2(\hat\kappa_+^2 - \hat\kappa_+^1) + \kappa_+^3(\hat\kappa_+^3 - \hat\kappa_+^1)$, as shown by the chain of arrows in Fig. 3. The operator $S(\kappa_+)$ can be visualised as a collection of three surfaces; S^1, S^2, S^3; plotted perpendicularly to this triangle in the base plane with values at the three edges being unity i.e., $S(\kappa_+^1 = 1, \kappa_+^2 = 0, \kappa_+^3 = 0) = 1$, $S(\kappa_+^1 = 0, \kappa_+^2 = 1, \kappa_+^3 = 0) = 1$, $S(\kappa_+^1 = 0, \kappa_+^2 = 0, \kappa_+^3 = 1) = 1$. A perspective view of the function S^1 is sketched in Fig. 4. For $\kappa_+^3 = 0$ the value of S^1 lies on the curve E_{12} and for $\kappa_+^2 = 0$ its value lies on the curve E_{13}. The problem is to construct the functions $S^1,\ S^2$ and S^3. This is now done in several steps.

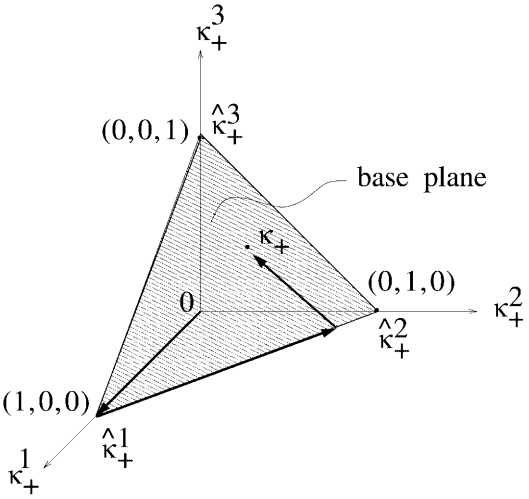

Fig. 3. Rectangular CARTESIAN coordinate system for κ_+^1, κ_+^2 and κ_+^3

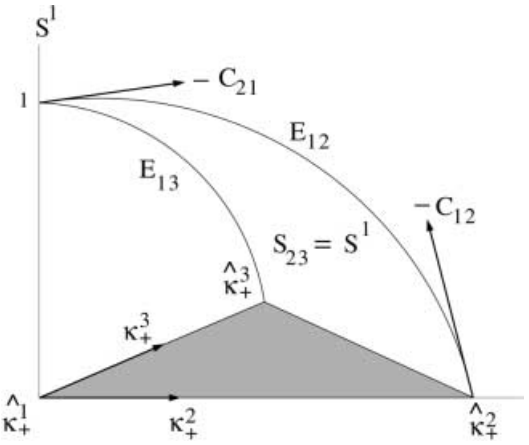

Fig. 4. Admissible triangular region on the base plane, the edge functions for the surface S^1

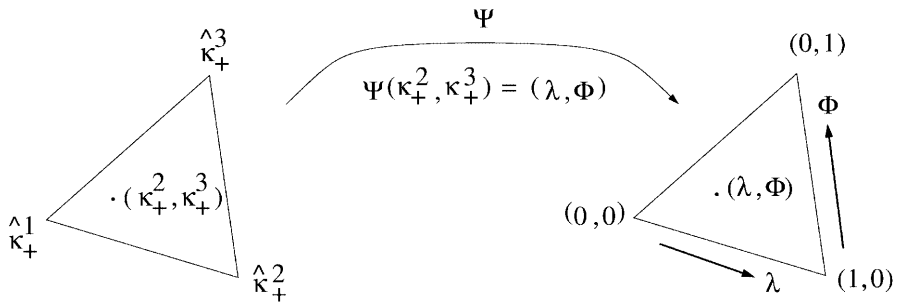

Fig. 5. Parameterisation of the domain triangle

3.2.1
Step I : Determination of the edge functions

Firstly, we determine the edges $E_{12}(\kappa_+^2)$ and $E_{13}(\kappa_+^3)$ of the surface S^1 as shown in Fig. 4. The *edge functions* $E_{ij}, 1 \le i,j \le 3,\ i \ne j$ are the analogue of S in PRIGOZHIN'S model as explained in §2.3. To determine $E_{12}(\kappa_+^2)$, the following four conditions are at our disposal:

$$E_{12}(0) = 1, \quad E_{12}(1) = 0,$$
$$\left.\frac{\partial E_{12}}{\partial \kappa_+^2}\right|_{\kappa_+^2=0} = -C_{21}, \quad \left.\frac{\partial E_{12}}{\partial \kappa_+^2}\right|_{\kappa_+^2=1} = -C_{12}\ . \tag{22}$$

The negative signs for the slopes express that E_{12} is monotonically decreasing with increasing κ_+^2. Similarly, the other edge function, $E_{13}(\kappa_+^3)$ should satisfy the conditions

$$E_{13}(0) = 1, \quad E_{13}(1) = 0,$$
$$\left.\frac{\partial E_{13}}{\partial \kappa_+^3}\right|_{\kappa_+^3=0} = -C_{31}, \quad \left.\frac{\partial E_{13}}{\partial \kappa_+^3}\right|_{\kappa_+^3=1} = -C_{13}\ . \tag{23}$$

There are many smooth edge functions satisfying these four conditions and providing monotonous functions. Quite a good fit to experimental data [2] is found if we use the mean value of a rational function and an ellipse the parameters of which are chosen to satisfy the boundary conditions:

$$E_{rat} = \frac{a+bt+ct^2}{d+t}, \quad E_{ell} = a - b\sqrt{(c-dt-t^2)},$$
$$\text{i.e. } E = \frac{1}{2}(E_{rat} + E_{ell})\ , \tag{24}$$

where a, b, c, d are some constants and t is a variable. It is clear that the first family represents *rational functions* and the second family represents *ellipses*. However, for the low concentration it is sufficient to consider only the family of rational functions.

3.2.2
Step II : Determination of the surface S^1

Having found these two edge functions we can solve the problem of finding the surface S^1. To obtain it we use the interpolation techniques, in particular the following special parameterisation. Consider the base plane triangle in Fig. 5. At the vertex $\hat\kappa_+^1$, both κ_+^2 and κ_+^3 are zero. Along the edge $\overline{\hat\kappa_+^1\hat\kappa_+^2}$, κ_+^2 is taken as a parameter and along the edge $\overline{\hat\kappa_+^1\hat\kappa_+^3}$, κ_+^3 is taken as a parameter. We want to transform this triangle into a new triangle with new paremeters λ and Φ (the meaning of which are clear from Fig. 5) so that the transformation is given by

$$\begin{aligned}(\kappa_+^2, \kappa_+^3) &= \lambda\,[(1,0) - (0,0)] + \Phi\,[\lambda(0,1) - \lambda(1,0)] \\ &= \lambda(1-\Phi, \Phi); \quad \lambda \in (0,1), \quad \Phi \in (0,1)\ .\end{aligned} \tag{25}$$

This implies

$$\begin{aligned}&\kappa_+^2 = \lambda(1-\Phi),\ \ \kappa_+^3 = \lambda\Phi,\ \ \kappa_+^2 = \lambda - \lambda\Phi = \lambda - \kappa_+^3, \\ &\implies \lambda = \kappa_+^2 + \kappa_+^3,\ \ \Phi = \frac{\kappa_+^3}{(\kappa_+^2 + \kappa_+^3)}\ .\end{aligned} \tag{26}$$

The surface $S_{23} = S^1$, as shown in Fig. 4, is now obtained on this new parametrised triangle by the formula

$$S_{23}(\lambda, \Phi) = (1-\Phi)E_{12}(\lambda) + \Phi E_{13}(\lambda)\ . \tag{27}$$

With the aid of (26), we have

$$\begin{aligned}S_{23} = S_{23}\left(\kappa_+^2, \kappa_+^3\right) &= \frac{1}{\left(\kappa_+^2 + \kappa_+^3\right)} \\ &\times \left[\kappa_+^2 E_{12}\left(\kappa_+^2 + \kappa_+^3\right) + \kappa_+^3 E_{13}\left(\kappa_+^2 + \kappa_+^3\right)\right]\ .\end{aligned} \tag{28}$$

Analogously, and simply by symmetry, the other component (surface) S_{13} having value unity at $\hat\kappa_+^2 = 1$ and value zero along the opposite edge is given by

$$\begin{aligned}S_{13} = S_{13}\left(\kappa_+^1, \kappa_+^3\right) &= \frac{1}{\left(\kappa_+^1 + \kappa_+^3\right)} \\ &\times \left[\kappa_+^1 E_{21}\left(\kappa_+^3 + \kappa_+^1\right) + \kappa_+^3 E_{23}\left(\kappa_+^3 + \kappa_+^1\right)\right]\ ,\end{aligned} \tag{29}$$

with analogous meanings of E_{21} and E_{23}. Finally, using the normality condition, the third surface is given by $S_{12} = 1 - S_{23} - S_{31}$. By using the normalisation $\kappa_+^1 + \kappa_+^2 + \kappa_+^3 = 1$, for simplicity, we rewrite these components of the operator S, so that they depend only on the two parameters κ_+^1 and κ_+^2, as follows:

$$\begin{aligned}S^1 = S_{23} = S(\kappa_+^1, \kappa_+^2) &= \frac{1}{(1-\kappa_+^1)} \\ &\times \left[\kappa_+^2 E_{12}(1-\kappa_+^1) + (1-\kappa_+^1-\kappa_+^2)E_{13}(1-\kappa_+^1)\right], \\ S^2 = S_{13} = S(\kappa_+^1, \kappa_+^2) &= \frac{1}{(1-\kappa_+^2)} \\ &\times \left[\kappa_+^1 E_{21}(1-\kappa_+^2) + (1-\kappa_+^1-\kappa_+^2)E_{23}(1-\kappa_+^2)\right]\ .\end{aligned} \tag{30}$$

Obviously, we have $S^3 = 1 - S^1 - S^2$. All the required components of the operator S are now determined. A prototype surface S^1 is represented by Fig. 6.

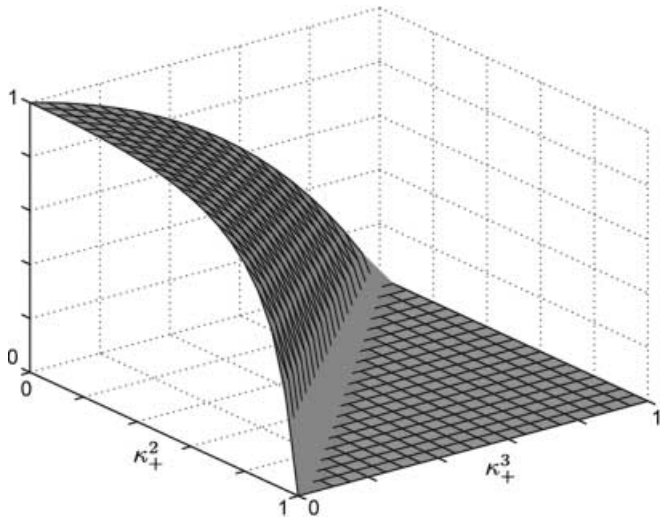

Fig. 6. The surface represented by S^1

3.3
Segregation for the flow on an inclined plane

Here we will develop a system of *ordinary differential equations* ($ODE's$) for κ_+^1, κ_+^2 and κ_+^3. The solution of this system will be used to find the free surface size segregation when the flow of a ternary mixture is considered on a plane that is inclined exactly at the angle of repose of the bulk material as shown in Fig. 1. With the operator S found in section §3.2, the differential equation (11) is now a system of ordinary differential equations that gives a model for particle size segregation. The corresponding initial value problem is

$$\frac{d}{dx}\begin{pmatrix}\kappa_+^1\\ \kappa_+^2\\ \kappa_+^3\end{pmatrix} = \frac{1}{x}\left[\begin{pmatrix}S^1\\ S^2\\ S^3\end{pmatrix} - \begin{pmatrix}\kappa_+^1\\ \kappa_+^2\\ \kappa_+^3\end{pmatrix}\right],$$

$$\begin{pmatrix}\kappa_+^1\\ \kappa_+^2\\ \kappa_+^3\end{pmatrix}(l) = \begin{pmatrix}\kappa_{+_0}^1\\ \kappa_{+_0}^2\\ \kappa_{+_0}^3\end{pmatrix} \quad , \tag{31}$$

where $\kappa_+^3 = 1 - \kappa_+^1 - \kappa_+^2$ and $S^3 = 1 - S^1 - S^2$. We tried to solve this *initial value problem* (IVP) analytically but we were unable to find any analytical solution. Therefore we solved the system numerically. Once $\kappa_+^i(x)$ are determined, $\kappa_s^i(x)$ can be found by the relation $\kappa_s^i = S^i$ and thereupon the spatial distribution of different species that corresponds to equation (13) of the PRIGOZHIN model

$$q^i(x) = \frac{\kappa_s^i(x)}{\int_0^l \kappa_s^i(x)dx} \quad . \tag{32}$$

A numerical simulation for $(\kappa_+^1, \kappa_+^2, \kappa_+^3)$ of the flux for the plane layer is represented in Fig. 7(a). We used the initial concentrations $\kappa_{+_0} = (\kappa_{+_0}^1, \kappa_{+_0}^2, \kappa_{+_0}^3) = (0.2, 0.3, 0.5)$ and the diameters of the grains $(d_1, d_2, d_3) = (1.0, 1.3, 1.5)$, i. e. for the diameter ratios (0.77, 0.87, 1.50). It is important to notice that these are the *only* two input parameters of this model.

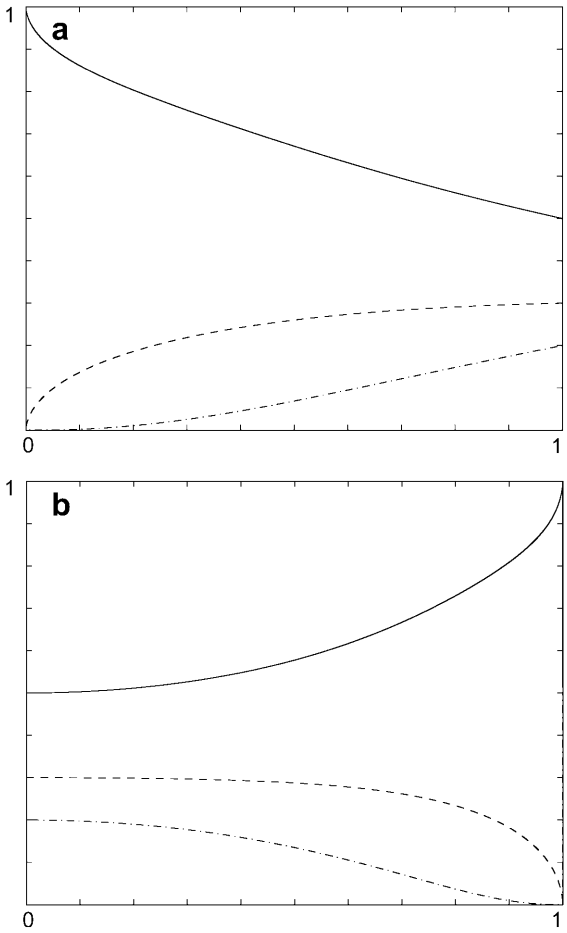

Fig. 7. Distribution of materials in the flux for different geometries

3.4
Segregation for the flow on the conical heap

It is known that there are many applications of conical pilings of granular materials in industries, science and technology as well as in the natural processes [3]. Solutions for a cone are needed, because there are some numerical solution schemes, e.g. [11], which consist of adding cones to the old pile. The pile itself need not be a cone. Now we want to derive a system of ordinary differential equations to model size segregation on the free surface of a conical heap formed by the granules themselves that is analogous to the system of ODE's for the plane (case) as discussed in §3.3. The geometry of the conical pile is such that α is the angle of repose of the bulk material, i.e. the angle between the base and the slant height of the pile, and R is the *radius* of its circular base. It is worthwhile to mention here that the parameter C defined by equation (16) depends only on the diameter ratio of the species and is independent of the geometry of the heap. Since $r = 0$ is a point of singularity, we will exclude this point from our consideration. This means, the point source lies outside the domain of computation, so that we can take $\omega = 0$ and $\omega_\Sigma = 0$ while dealing with the differential equations. For the case of the cone we have axial symmetry, so that $k = 1$. Consequently, equations (7) and (8) reduce to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rm\frac{\partial h}{\partial r}\right) = \rho\dot{h} \ , \tag{33}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\kappa_+ r m \frac{\partial h}{\partial r}\right) = \kappa_s \rho \dot{h} \ , \tag{34}$$

respectively. Integrating equation (33) from $r > 0$ to $r \leq R$ we have

$$-rm\frac{\partial h}{\partial r} = \left(\frac{R^2 - r^2}{2}\right)\rho\dot{h} \ , \tag{35}$$

since $h = 0$ for $r = R$. Applying the product rule of differentiation for equation (34) and using the results of equations (33) and (35) and (9) we obtain the following differential equation to model the size segregation on the surface of a conical pile of granular materials that is the analogue of equation (11) for the plane, namely

$$\frac{d\kappa_+}{dr} = \frac{2r}{r^2 - R^2}\left[S(\kappa_+) - \kappa_+\right], \quad \kappa_+(0) = \kappa_0 \ . \tag{36}$$

Thus, for the ternary mixture, the required system of *ODE's* is

$$\frac{d}{dx}\begin{pmatrix}\kappa_+^1\\ \kappa_+^2\\ \kappa_+^3\end{pmatrix} = \frac{2r}{r^2 - R^2}\left[\begin{pmatrix}S^1\\ S^2\\ S^3\end{pmatrix} - \begin{pmatrix}\kappa_+^1\\ \kappa_+^2\\ \kappa_+^3\end{pmatrix}\right],$$

$$\begin{pmatrix}\kappa_+^1\\ \kappa_+^2\\ \kappa_+^3\end{pmatrix}(l) = \begin{pmatrix}\kappa_{+_0}^1\\ \kappa_{+_0}^2\\ \kappa_{+_0}^3\end{pmatrix} \ . \tag{37}$$

It should be mentioned here that for the plane case, the source is above the point where $x = l$, and for the cone it is taken to be at the center of the cone, i.e. at $r = 0$. That can also be seen from corresponding initial data. Using the data of §3.3 for the diameter ratios and initial concentrations, the distribution of the three different species in the flux on the surface of the cone are represented in Fig. 7(b) which is the analogue of Fig. 7(a) for the plane. Here, the horizontal and vertical axes represent the position and the distribution of materials in the flux, respectively. Both of these figures represent the distribution of materials in the flux with diameter ratios 0.77, 0.87 and 1.50 and initial concentrations 20%, 30% and 50%. Figures (a) and (b), respectively, correspond to plane and conical granular bed. And the legend for the both figures is: (. _ . _ . _ .) for κ_+^1, (− − −) for κ_+^2 and (_____) for κ_+^3, respectively. These figures also show an interesting interaction between the concentrations and the diameter ratios. These numerical simulations indicate the fact that our new and extended model is quite able to reflect the physical phenomena of free surface size segregation in polydisperse granular material.

4
Comparison of our result with Prigozhin's result

We are interested in free surface size segregation of granular materials. A particular case of our model reduces to the model of Prigozhin (by setting $\kappa_+^3 = 0$) with slightly a different operator S. Our main goal of investigation is to present a simple and appropriate model for the case of mixtures consisting of more than two species. For the interpolation of the material concentrations of different species in the flow and in the new heap surface layer Prigozhin used the pair of parametrised quadratic curves, given in (19). However, there is no simple way of generalising this approach to three or more species. Instead we simply use the *algebraic* function, given in (24). We need only one edge function to determine the density of the spatial distribution of different grain sizes if we want to deduce the result of Prigozhin from our model for a binary mixture. One of the most important facts is that our model can be easily generalised to more than three species by using simplices of dimension $n - 1$ insteat of the triangle, if the domain of the operator S is of dimension n. Of course, we should use the same approach of parameterisation for the tetrahedron and use the previously found surfaces (or similar surfaces) S^1, S^2 and S^3 etc. on each face of the tetrahedron. The remaining procedure is exactly the same. The following figure, Fig. 8, gives the direct and quantitative comparison between our results and those of Prigozhin and measured data of Drahun and Bridgwater [2] for the flow of a binary mixture on an inclined plane at the angle of repose. This comparison justifies our arguments and shows that our model is sufficiently accurate for the study of the size segregation. These figures represent size segregation of a binary mixture for various fed compositions and diameter ratios. They represent the distribution of the first components. The distribution of the second components can easily be found from normalisation. In all these figures, the solid lines represent our theoretical results, the dotted lines represent the theoretical result of Prigozhin [13] and the stars represent the measured data of Drahun and Bridgwater [2]. The horizontal and vertical axes represent the position, x/l, of the heap and the density of the spatial distribution , $q(x/l)$, of materials, respectively.

5
Experiments

As we have already mentioned, no measurements were available for mixtures with a large number of constituents for free surface size segregation. So, to validate and check the quality of the new and extended model, we conducted our own experiments. Prigozhin [13] also pointed out that the experiments should be repeated at least to check the quality of the previous model. We constructed an apparatus similar to that of Drahun and Bridgwater [2] and performed the experiments similarly. The picture of the apparatus is presented in Fig. 9.

5.1
Experimental Set Up

5.1.1
Circumstances and procedure

Experiments were performed for an *inclined plane*. The temperature and humidity of the laboratory were about 21^0C and 70%, respectively. The length, width and the height of the apparatus (inner free space without walls) were 47.8 cm, 9.9 cm and 42.0 cm, respectively. A *spoon* of volume 48 cm^3 of the length equal to the inner width of the apparatus was used to feed the material from the top of the plane inside the apparatus. The granules used

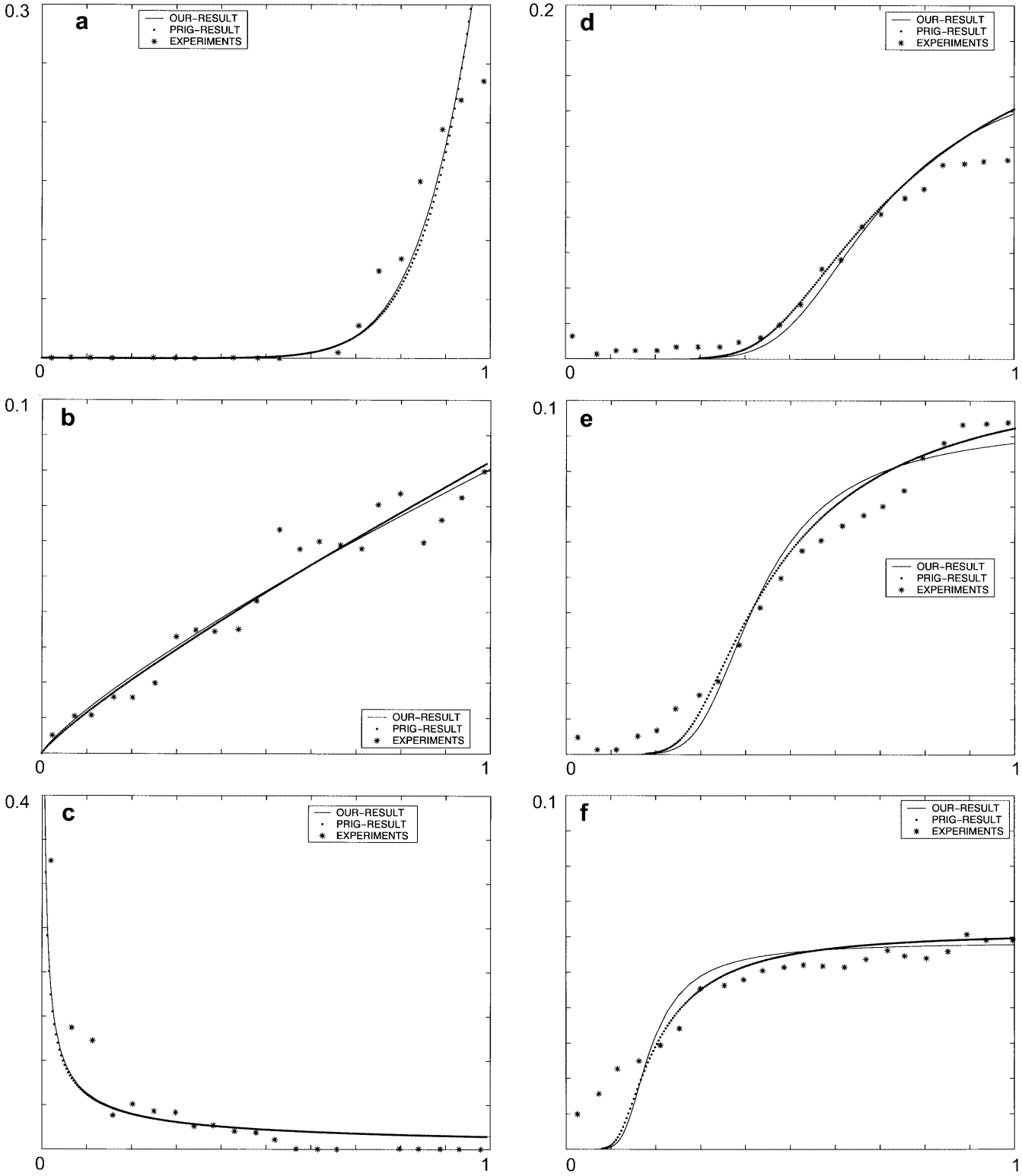

Fig. 8a–f. Density of spatial distribution of materials. The data used for initial concentrations and diameter ratios are: **a** 1%, 0.5; **b** 1%, 0.82 ; **c** 1%, 1.28; **d** 22.4%, 0.5; **e** 44.8%, 0.5 and **f** 72.7%, 0.5, respectively

in the experiments were *spherical glass balls* of different diameters d_1, d_2, and d_3, in the ranges (0.75, 0.8), (0.8, 1.0) and (1.0, 1.12) mm, respectively. The concentrations of the materials of diameters d_1, d_2, and d_3 were taken to be $\kappa^1_{+_0} = 0.1428$, $\kappa^2_{+_0} = 0.5715$ and $\kappa^3_{+_0} = 0.2857$, and it was assumed that the *diameters are uniformly distributed* in their respective intervals. The *basal plane* was coated with *sand papers* to guarantee that the *bed friction angle*

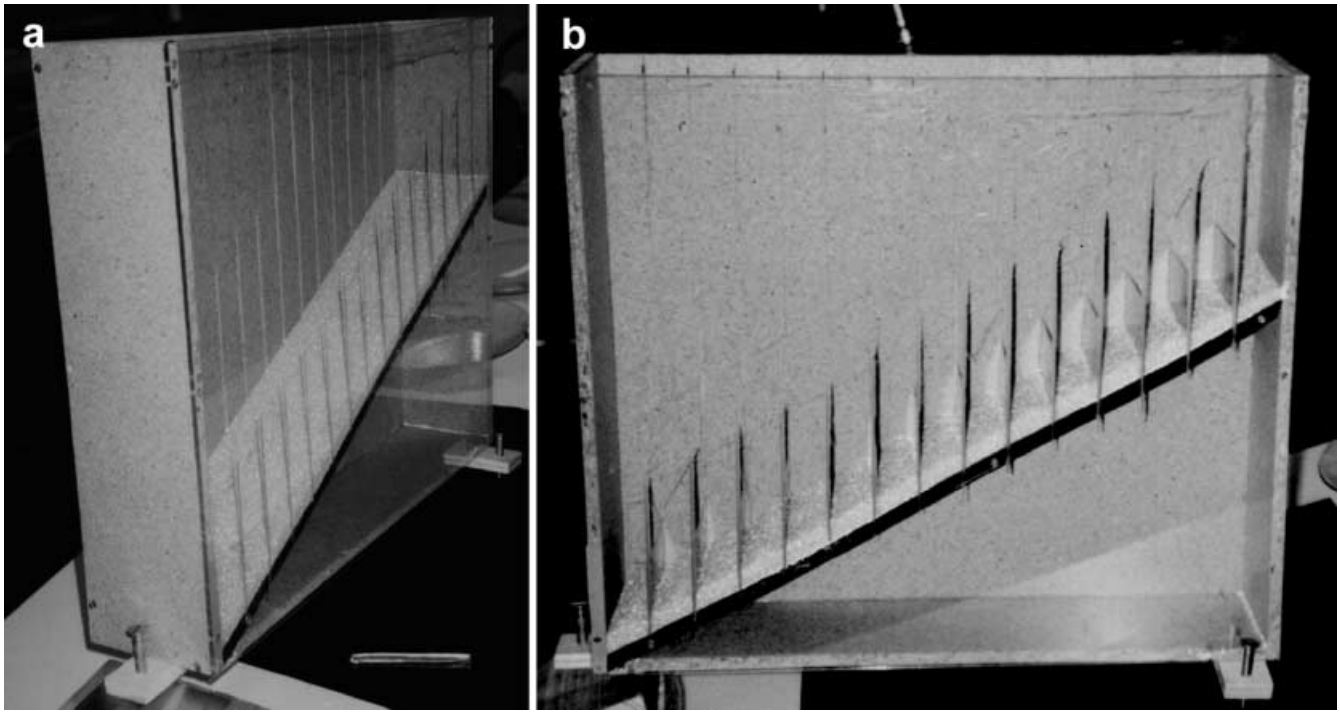

Fig. 9. The apparatus used for the experiments

is larger than the angle of repose of the glass beads. The inclined plane first was fixed at the *angle of repose* of the glass beads, α approximately 20°. Then, a *monolayer* was formed on the surface of the inclined plane by the material of diameters d_2. Afterwards, material was fed from the top end of the inclined plane, almost from zero height, proportional to their amount of concentrations in the whole bulk material. The complete chain of feeding (we will call it a *loop*) was as follows: firstly, we discharged 4 *spoons* of material of diameter d_2, followed by 2 *spoons* of material of diameter d_3 and finally, we discharged 1 *spoon* of material of diameter d_1. The process was repeated until the whole material was finished.

5.1.2 Avalanches

Although the material is continuously poured on the top of the pile, it does not flow immediately down the face because of the difference between the static and dynamic *internal friction angles*. Once the static angle is exceeded the material flows down the face of the pile [3], [7], [8]. Many small *avalanches* were observed during the process of filling and bed formation. After each loop a small avalanche was artificially produced by a small but controlled vibration on the apparatus. The reason for this was to control the segregation and *rule out* the effect of large avalanches. Otherwise, large avalanches might have occurred and destroyed the mechanism of free surface flow that would violate some of the most important assumptions of the theory. During the flow it was observed that the largest particles were easily rolling; they moved quickly down the plane since only small voids exist on the free surface. On the other hand, the smallest particles were less prone to roll down on the surface because they could find sufficient voids to be trapped. The behaviour of the medium size particles was obviously in between. Kinetic sieving is responsible for this, see [3]. Before an avalanche was triggered, the angle of repose of the bulk just below the charging point was larger, but after the avalanche had passed it became less at that point. This is so because the material can flow only when the heap exceeds the critical angle of repose. However, since our material consists of spherical glass beads with *smooth surfaces*, the variation of the local angle of repose along the free surface was very small.

5.1.3 Separation of the materials and density of spatial distribution

Upon the termination of the feeding, the free surface plane was kept horizontally *to avoid bulk motion and segregation* during analysis. The whole plane was divided into fifteen equal parts. Thin rectangular steel plates, that fit exactly inside the box, were pushed into the slots set in the perspex walls thereby isolating sections of the bed. The solids between the plates were carefully withdrawn from the *far end*, because the apparatus was made such that the far end vertical wall could be removed. The mixtures between the plates were analysed by *sieving*. We sieved it by using the sieves of sizes 0.8 mm and 1.0 mm of diameters, respectively, to separate materials of different diameter sizes. We measured the separated material by *weighing*. Finally, from the knowledge of the masses of a component found in the various boxes, the *spatial probability distribution*, q, is calculated.

5.1.4 Loss of material and non-uniformity of the heap

The amount of the material lost was 1.9% of the total fed materials. The material was lost during the *feeding*

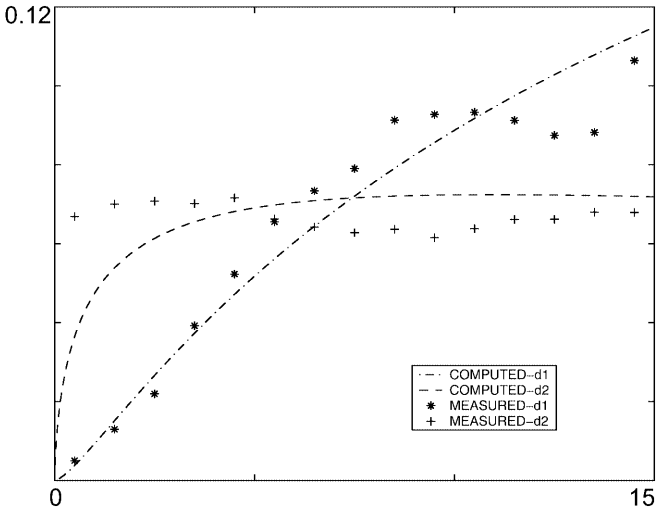

Fig. 10. Density of spatial distribution of materials with initial concentrations 14.28% and 57.15% and diameter ratios 0.86, 0.85 and 1.37

process, when *reclaiming* it from the separated portions of the apparatus and mainly during the *filtration*. Some glass balls were *sticking in the net* of the sieves during the filtration. Such a loss was higher in the net of size 1.0 mm than in 0.8 mm. The amount of loss was larger during the sieving of the first few portions from the far end part of the plane. We also observed the *transition* of the materials from one interval to another due to the *uniform distribution* of the material in their respective interval of diameters. A variation of mass in each section was detected from the measurements. Less amount of mass was deposited just below and at the far end of the discharge point and a relatively high amount of material was deposited at the middle sections. This means that the height of the (total) material bed, parallel to the plane, was not exactly constant. We performed experiments twice. Both of the measurements were almost the same, and we took the *mean value* of these measurements. The result presented here (in §6, Fig. 10) is for that mean value.

6 Comparison of the Measured Data with the New Model

Comparison of the measured data with our model for the material with diameters d_1 and d_2 is presented in Fig. 10. Here, the horizontal and vertical axes represent the position and spatial distribution of materials, respectively. For this purpose, we took as edge function a *rational* function, which is found to be better in this case. Using the normalisation, the corresponding result for d_3 can be easily derived. During the experiment we noticed many sources of errors. The figure reveals that our model is able to analyse the qualitative behaviour of size segregation in granular materials when the mixture consists of three materials of different sizes. The most important fact is that we were able to formulate and develop a new model and produce data by reliable laboratory experiments for a mixture of three materials and analyse the quantitative and qualitative correlation between the data and the continuum model.

7 Conclusion

A mathematical model of free surface size segregation of granular matter has been proposed. It consists of a set of ordinary differential equations predicting how the concentrations of particles with different diameters change with the distance from the source. Its core is a model describing how the deposition rates depend on the concentrations and the diameter ratios of the species in the flux. Following Prigozhin we interpolate values which are known from experiments by Drahun and Bridgwater where one diameter is predominant. However, our interpolation scheme is simpler, i.e. it manages without numerically solving an implicit equation and, more importantly, it may be applied to more than two species, thus satisfying an industrial need. Numerical simulations were performed for conical and ramp-like geometries and for binary and ternary mixtures. Since we did not find any data on ternary mixtures in the literature we performed our own experiments. For binary mixtures the new model turned out to be as accurate as that by Prigozhin and the agreement for ternary mixtures is fair. A final quantitative estimation of the quality of the model is nevertheless somewhat difficult:

(1) Many more experiments are needed to check, whether the mean deviation between measurement and theory is of the same order of magnitude as natural fluctuations between subsequent experiments, or else there are systematic errors.

(2) As we are no experts in experimental process engineering the experimental data themselves might not be sufficiently reliable. We consider this as an incentive to further experiments on mixtures for more than two species.

(3) Finally, in reality there is not a discrete distribution, but a whole continuum of diameters. To apply our model we divided this continuum into three intervals and treated the particles belonging to one interval as if all of them had the same diameter. The effect of this simplification has not yet been investigated.

May this be as it is, if we consider the shape of the source, its composition, and the ratio of the involved diameters to be the only input parameters and, moreover, if the model is sufficiently simple to be implemented, for instance, into an industrial controller, the agreement seems to be astonishingly good.

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